Geometric Algorithms for Zero Cancellation with Application to Unknown-State, Unknown-Input Reconstruction

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Abstract: This work deals with zero cancellation in discrete-time linear multivariable systems and its application to unknown-state unknown-input reconstruction. The problem is first tackled by considering a scheme based on the use of a feedforward compensator, then an equivalent cascade filter is obtained by means of duality arguments. The precompensator devised preserves crucial properties of the original system like reachability and right-invertibility, while the cascade filter maintains observability and left-invertibility and, therefore, is the appropriate solution to be exploited in connection with state and input reconstructors. The design method relies on the geometric approach. Two solutions, ensuing from different choices of the resolving subspace, are examined for the feedforward compensator and, consequently, its dual counterpart: the most convenient for its impact on reconstruction is then pursued.

Keywords: Zero cancellation; state and input reconstruction; geometric approach.

1. INTRODUCTION

As is well-known, the solution to the zero cancellation problem in linear multivariable systems is nonunique in general. Hence, the problem lends itself to various formulations, that differ for the further specifications. A report on the state of the art of pole-zero cancellation in the middle nineties can be found in Douglas and Athans (1996). The zero displacement problem in terms of transfer function matrices and corresponding realizations is deeply investigated in Oară and Andrei (2009): the solutions provided in that paper also feature properties like having the minimum McMillan degree or being J-unitary or J-inner. The design of a precompensator for cancelling finite, possibly multiple, invariant zeros in strictly-proper, continuous-time systems is discussed in Wan et al. (2009): the solution proposed in that work also aims at maintaining stabilizability, right-invertibility, and left-invertibility of the original system in the equivalent cascade.

Zero cancellation is first considered in connection with reconstruction of unknown inputs, in the presence of unknown initial states, in Marro and Zattoni (2010). That article is focused on strictly-proper, discrete-time systems which are also reachable, observable, and left-invertible. The filter for zero cancellation is obtained as the dual counterpart of a feedforward compensator whose structure replicates not only that of the minimum-phase invariant zeros, but also that of the internal eigenvalues of the constrained reachability subspace of the original system. In Marro et al. (2010), similar issues are tackled for non-strictly-proper, discrete-time systems, which are still assumed to be in minimal form and left-invertible. However, the mechanism for zero cancellation proposed in Marro et al. (2010) hinges on the determination of the structure of the sole minimum-phase invariant zeros, through matrices which, once reproduced in the filter, guarantee its minimal dynamic order. In Marro and Zattoni (2011), the technique for zero cancellation introduced in Marro et al. (2010) — namely, the one based on the minimal resolving structure — is considered independently of unknown-state, unknown-input reconstruction and, therefore, can be extended to non-strictly-proper, discrete-time systems which are still in minimal form, but are not required to be left-invertible (or right-invertible, in the dual context) anymore.

In this work, zero cancellation in systems which, in general, are non-strictly-proper, non-minimal, non-right-invertible, and non-left-invertible is approached by extending the methodology first presented in Marro and Zattoni (2010): namely, the method based on the replica of the structure of all the internal stabilizable eigenvalues of the maximal output-nulling controlled-invariant subspace of the original system. The comparison with the extension to nonminimal realizations of the solution discussed in Marro and Zattoni (2011) shows that the filter obtained as the dual counterpart of the feedforward compensator including the structure of all the internal assignable eigenvalues of the maximal output-nulling controlled-invariant subspace is more convenient in the applications to unknown-state unknown-input reconstruction, since it ensures reconstruction in the minimum number of steps, provided that the filter be permanently connected to the original system.

Notation: $\mathbb{Z}$ and $\mathbb{R}$ stand for the sets of integer and real numbers, respectively. $\mathbb{C}^{\odot}$, $\mathbb{C}^{\otimes}$, and $\mathbb{C}^{\circ}$ respectively stand for the open unit disc, the open set outside the unit disc, and the unit circle in the complex plane $\mathbb{C}$. Matrices and linear maps are denoted by capital letters, like A. The spectrum, the image, and the kernel of A are denoted by $\sigma(A)$, $\operatorname{im} A$, and $\ker A$, respectively. The restriction of a linear map A to an A-invariant subspace $\mathcal{J}$ is denoted by $A|_{\mathcal{J}}$. The quotient space of a vector space $\mathcal{X}$ over a subspace $\mathcal{V} \subseteq \mathcal{X}$ is denoted by $\mathcal{X}/\mathcal{V}$. The orthogonal complement of $\mathcal{V}$ is denoted by $\mathcal{V}^{\perp}$. The inverse image of $\mathcal{V}$ through a linear map B is denoted by $B^{-1}\mathcal{V}$. The dimension of $\mathcal{V}$ is denoted by $\dim \mathcal{V}$. The symbols I_n and $O_{m\times n}$ are respectively used for the identity matrix of dimension n and the $m \times n$ zero matrix (subscripts are omitted when the dimensions are clear from the context). The symbol $\operatorname{diag}\{M_1, M_2, \ldots, M_r\}$ stands for a block diagonal matrix whose blocks on the main diagonal are M_1, M_2, ..., M_r, in order.

2. ZERO CANCELLATION BY A FEEDFORWARD COMPENSATOR: PROBLEM STATEMENT

The discrete-time linear time-invariant system

$$x_{t+1} = A\,x_t + B\,u_t, \tag{1}$$

$$y_t = C\,x_t + D\,u_t, \tag{2}$$

is considered, where $t \in \mathbb{Z}$ is the time variable and $x \in \mathcal{X} = \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^q$, with $p,\, q \le n$, are the state, the input, the output, respectively. A, B, C, D are constant real matrices of appropriate dimensions. Without loss of generality, $\left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right]$ and $[C\ D]$ are assumed to be of full-rank. Let system (1), (2) satisfy the following assumption:

$\mathcal{A}\,1$. $\min \mathcal{J}(A, \mathcal{B}) = \mathcal{X}$.

Assumption $\mathcal{A}\,1$ (i.e., reachability of the original system) is a standard assumption in control problems, since it allows stabilization. Indeed, the weaker assumption that the subspace $\min \mathcal{J}(A, \mathcal{B})$ be externally stable would be sufficient to this aim. However, we will refer to $\mathcal{A}\,1$, since in the application to unknown-state, unknown-input reconstruction of the dual problem of zero cancellation by a cascaded filter (Section 7), the assumption of observability of the system is mandatory.

The problem of zero cancellation by a feedforward compensator is formulated as the problem of synthesizing a feedforward compensator

$$x_{f,t+1} = A_f\,x_{f,t} + B_f\,w_t, \tag{3}$$

$$u_t = C_f\,x_{f,t} + D_f\,w_t, \tag{4}$$

such that the cascade

$$x_{e,t+1} = A_e\,x_{e,t} + B_e\,w_t, \tag{5}$$

$$y_t = C_e\,x_{e,t} + D_e\,w_t, \tag{6}$$

of (3), (4) and the original system (1), (2) satisfy the following conditions:

$\mathcal{C}\,1$. $\min \mathcal{J}(A_e, \mathcal{B}_e) = \mathcal{X}_e$;

$\mathcal{C}\,2$. $\mathcal{Z}(A_e, B_e, C_e, D_e) = \mathcal{Z}_{NMP}(A, B, C, D) \cup \mathcal{Z}^{\circ}(A, B, C, D)$;

$\mathcal{C}\,3$. $\max \mathcal{V}(A, B, C, D) + \min \mathcal{S}(A, B, C, D) = \mathcal{X} \implies \max \mathcal{V}(A_e, B_e, C_e, D_e) + \min \mathcal{S}(A_e, B_e, C_e, D_e) = \mathcal{X}_e$.

Condition $\mathcal{C}\,1$ states the requirement that reachability be preserved in the cascade system.

Condition $\mathcal{C}\,2$ expresses the requirement that all the minimum-phase invariant zeros of the original system be cancelled by the feedforward compensator, so that the invariant zeros of the cascade result to be the union of the nonminimum-phase invariant zeros and the invariant zeros on the unit circle. Indeed, this specification can be refined in order for the feedforward compensator to cancel appropriate subsets of the minimum-phase invariant zeros of the original system. How these subsets must be selected to guarantee that the problem be solvable will be shown in Remark 18.

Condition $\mathcal{C}\,3$ deals with the property of right-invertibility and states the requirement that this property be retained by the cascade when it belongs to the original system. It is worth mentioning that right-invertibility plays a key role in control problems like, e.g., reference tracking and signal decoupling.

3. SYNTHESIS OF THE FEEDFORWARD COMPENSATOR: SIMILARITY TRANSFORMATIONS

In this section, a sequence of three similarity transformations leads to a state-space representation revealing the structure of the invariant zeros and thus allowing a straightforward determination of the key subspace for the synthesis of the feedforward compensator. Indeed, two different choices for the resolving subspace are possible, as will be clear from Sections 4 and 5. Consequently, the resulting devices will guarantee different performances, with an impact which is particularly evident when they are considered in the dual context and applied to unknown-state, unknown-input reconstruction, as will be shown in Section 7.

The reader is referred to Marro and Zattoni (2011) for the proofs of the following lemmas.

Lemma 1. Consider system (1), (2). Let the linear map F be such that $(A + BF)\,\mathcal{V}^* \subseteq \mathcal{V}^*$ and $\mathcal{V}^* \subseteq \ker(C + DF)$. Perform the similarity transformation $T = [T_1\ T_2\ T_3\ T_4]$, where $\operatorname{im} T_1 = \mathcal{R}_{\mathcal{V}^*}$, $\operatorname{im}\,[T_1\ T_2] = \mathcal{V}^*$, $\operatorname{im}\,[T_1\ T_3] = \mathcal{S}^*$. Then,

$$A'_F = T^{-1}(A + BF)\,T = \begin{bmatrix} A'_{11} & A'_{12} & A'_{13} & A'_{14} \\ O & A'_{22} & A'_{23} & A'_{24} \\ O & O & A'_{33} & A'_{34} \\ O & O & A'_{43} & A'_{44} \end{bmatrix}, \tag{7}$$

$$C'_F = (C + DF)\,T = \begin{bmatrix} O & O & C'_3 & C'_4 \end{bmatrix}. \tag{8}$$

Remark 2. The set of the internal eigenvalues of $\mathcal{R}_{\mathcal{V}^*}$ is equal to the set of the eigenvalues of A'_{11}: i.e., $\sigma((A + BF)|_{\mathcal{R}_{\mathcal{V}^*}}) = \sigma(A'_{11})$. The set of the internal unassignable eigenvalues of $\mathcal{V}^*$, or, equivalently, the set of the invariant zeros of system (1), (2), is equal to the set of the eigenvalues of A'_{22}: i.e., $\sigma((A + BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \mathcal{Z}(A, B, C, D) = \sigma(A'_{22})$.

Lemma 3. Consider system (1), (2). Let the linear map F be such that $(A + BF)\mathcal{V}^* \subseteq \mathcal{V}^*$, $\mathcal{V}^* \subseteq \ker(C + DF)$, and $\sigma((A + BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cap \sigma((A + BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \emptyset$. Refer to (7), (8) and perform the similarity transformation $T' = [T'_1\ T''_2\ T'_3\ T'_4]$, where T'_1, T''_2, T'_3, T'_4, are defined by

$$T_1' = \begin{bmatrix} I_{n_1} \\ O \\ O \\ O \end{bmatrix},\ T_2'' = \begin{bmatrix} X \\ I_{n_2} \\ O \\ O \end{bmatrix},\ T_3' = \begin{bmatrix} O \\ O \\ I_{n_3} \\ O \end{bmatrix},\ T_4' = \begin{bmatrix} O \\ O \\ O \\ I_{n_4} \end{bmatrix}. \tag{9}$$

with $n_1 = \dim \mathcal{R}_{\mathcal{V}^*}$, $n_2 = \dim \mathcal{V}^* - n_1$, $n_3 = \dim \mathcal{S}^* - n_1$, $n_4 = n - (n_1 + n_2 + n_3)$, and with X denoting the solution of the Sylvester equation $A_{11}'X - XA_{22}' = -A_{12}'$. Then,

$$A_F'' = T'^{-1} A_F' T' = \begin{bmatrix} A_{11}' & O & A_{13}'' & A_{14}'' \\ O & A_{22}' & A_{23}' & A_{24}' \\ O & O & A_{33}' & A_{34}' \\ O & O & A_{43}' & A_{44}' \end{bmatrix}, \tag{10}$$

$$C_F'' = C_F' T' = C_F'. \tag{11}$$

Lemma 4. Consider system (1), (2). Let the linear map F be such that $(A+BF)\,\mathcal{V}^* \subseteq \mathcal{V}^*$, $\mathcal{V}^* \subseteq \ker(C+DF)$, and $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cap \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \emptyset$. Refer to (10), (11). Let $\mathcal{J}_S = \operatorname{im} J_S$, $\mathcal{J}_0 = \operatorname{im} J_0$, and $\mathcal{J}_U = \operatorname{im} J_U$, with J_S, J_0, and J_U full-rank matrices, be the maximal A_{22}'-invariant subspaces such that $\sigma(A_{22}'|_{\mathcal{J}_S}) \subset \mathbb{C}^{\odot}$, $\sigma(A_{22}'|_{\mathcal{J}_0}) \subset \mathbb{C}^{\circ}$, and $\sigma(A_{22}'|_{\mathcal{J}_U}) \subset \mathbb{C}^{\otimes}$, respectively. Perform the similarity transformation $T'' = [T_1'\ T_2'''\ T_3'\ T_4']$, where T_1', T_3', T_4' are defined by (9) and

$$T_2''' = \left[\begin{array}{ccc} O & O & O \\ \hline J_S & J_0 & J_U \\ \hline O & O & O \\ \hline O & O & O \end{array}\right].$$

Then,

$$A_F''' = T''^{-1} A_F'' T'' = \left[\begin{array}{c|ccc|c|c} A_{11}' & O & O & O & A_{13}'' & A_{14}'' \\ \hline O & A_{22_S}'' & O & O & A_{23_S}'' & A_{24_S}'' \\ O & O & A_{22_0}'' & O & A_{23_0}'' & A_{24_0}'' \\ O & O & O & A_{22_U}'' & A_{23_U}'' & A_{24_U}'' \\ \hline O & O & O & O & A_{33}' & A_{34}' \\ \hline O & O & O & O & A_{43}' & A_{44}' \end{array}\right], \tag{12}$$

$$C_F''' = C_F'' T'' = \left[\, O | O\ O\ O | C_3' | C_4' \,\right]. \tag{13}$$

Remark 5. The set of the internal unassignable eigenvalues of $\mathcal{V}^*$ in the open unit disc of the complex plane or, equivalently, the set of the minimum-phase invariant zeros of (1), (2) is equal to the set of the eigenvalues of A_{22_S}'': i.e., $\mathcal{Z}_{MP}(A,B,C,D) = \sigma(A_{22_S}'')$. The set of the internal unassignable eigenvalues of $\mathcal{V}^*$ on the unit circle or, equivalently, the set of the invariant zeros of (1), (2) with unit absolute value is equal to the set of the eigenvalues of A_{22_0}'': i.e., $\mathcal{Z}_0(A,B,C,D) = \sigma(A_{22_0}'')$. The set of the internal unassignable eigenvalues of $\mathcal{V}^*$ in the open set outside the unit disc or, equivalently, the set of the nonminimum-phase invariant zeros of (1), (2) is equal to the set of the eigenvalues of A_{22_U}'': i.e., $\mathcal{Z}_{NMP}(A,B,C,D) = \sigma(A_{22_U}'')$.

4. SYNTHESIS OF THE FEEDFORWARD COMPENSATOR: $\mathcal{V}_S^*$ AS KEY SUBSPACE

The representation of the system obtained at the end of the previous section allows us to easily determine a subspace which is associated with the structure of the minimum-phase invariant zeros of the system and, therefore, can play a crucial role in the synthesis of the feedforward compensator. As will be shown in the following, the subspace inherently connected to the minimum-phase invariant zeros of the system is an output-nulling controlled-invariant subspace whose internal eigenvalues match the minimum-phase invariant zeros of the system. Theorem 6 formally introduces the abovementioned subspace, henceforth denoted by $\mathcal{V}_S^*$. Corollary 7 relates $\mathcal{V}_S^*$ with the linear maps which can be used in the synthesis of the precompensator. Proofs are omitted. Indeed, they develop along the same lines of those in Marro and Zattoni (2011), except for the fact that minimal realizations are not assumed in this work.

Theorem 6. Consider system (1), (2). Let the linear map F be such that $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$, $\mathcal{V}^* \subseteq \ker(C+DF)$, and $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cap \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \emptyset$. Refer to (12), (13) and let the subspace $\mathcal{V}_S^*$ be defined by

$$\mathcal{V}_S^* = \operatorname{im} V_S^{*\prime\prime\prime} = \operatorname{im} \left[\, O | I_{n_S}\ O\ O | O | O \,\right]^\top, \tag{14}$$

where $n_S = \dim \mathcal{J}_S$. Then,

- (i) $\mathcal{V}_S^*$ is an output-nulling controlled-invariant subspace of (1), (2);
- (ii) the set of the internal eigenvalues of $\mathcal{V}_S^*$ is equal to the set of the minimum-phase invariant zeros of (1), (2): i.e., $\sigma((A+BF)|_{\mathcal{V}_S^*}) = \mathcal{Z}_{MP}(A,B,C,D)$.

Corollary 7. Consider system (1), (2). Let the linear map F be such that $(A+BF)\,\mathcal{V}^* \subseteq \mathcal{V}^*$, $\mathcal{V}^* \subseteq \ker(C+DF)$, and $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cap \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \emptyset$. Let $\bar{T} = TT'T''$ where T, T', T'' were introduced in Lemmas 1 to 4. Let $V_S^* = \bar{T} V_S^{*\prime\prime\prime}$, where $V_S^{*\prime\prime\prime}$ was introduced in Theorem 6. Then,

$$AV_S^* - V_S^* W_S = -BL_S, \tag{15}$$

$$CV_S^* = -DL_S, \tag{16}$$

hold with $W_S = A_{22_S}''$ and $L_S = FV_S^*$.

Remark 8. The subspace $\mathcal{V}_S^*$ is the minimal output-nulling controlled-invariant subspace whose internal eigenvalues are all unassignable and contained in the open unit disc. Hence, the use of $\mathcal{V}_S^*$ with the associated linear maps W_S and L_S in the synthesis of the feedforward compensator leads to the minimal-order feedforward compensator.

5. SYNTHESIS OF THE FEEDFORWARD COMPENSATOR: $\mathcal{V}_G^*$ AS KEY SUBSPACE

Although the subspace $\mathcal{V}_S^*$, introduced in Section 4, is the subspace more directly connected with the structure of the minimum-phase invariant zeros of the original system, a different subspace is more profitably employed for zero cancellation when the final application concerns reconstruction of unknown inputs in the presence of unknown initial states. The subspace at issue is the maximal internally-stabilizable output-nulling controlled-invariant subspace of the original system, henceforth denoted by $\mathcal{V}_G^*$. The set of the internal eigenvalues of $\mathcal{V}_G^*$ also includes the internal eigenvalues of the subspace $\mathcal{R}_{\mathcal{V}^*}$, since $\mathcal{R}_{\mathcal{V}^*} \subseteq \mathcal{V}_G^*$. Those eigenvalues are all assignable (see Remark 2). Therefore, the linear map F should also guarantee internal stability of $\mathcal{V}_G^*$ by assigning all the eigenvalues of $\mathcal{R}_{\mathcal{V}^*}$ in the open unit disc: i.e.,

$$\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \subset \mathbb{C}^{\odot}. \tag{17}$$

Apart from this further property of F, this section follows the lines of Section 4.

Theorem 9. Consider system (1), (2). Let the linear map F be such that $(A+BF)\mathcal{V}^* \subseteq \mathcal{V}^*$, $\mathcal{V}^* \subseteq \ker(C+DF)$, $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cap \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \emptyset$, and $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \subset \mathbb{C}^{\odot}$. Refer to (12), (13) and let the subspace $\mathcal{V}_G^*$ be defined by

$$\mathcal{V}_G^* = \operatorname{im} V_G^{*\prime\prime\prime} = \operatorname{im} \left[\begin{array}{cc} I_{n_1} & O \\ \hline O & I_{n_S} \\ O & O \\ O & O \\ \hline O & O \\ \hline O & O \end{array}\right]. \tag{18}$$

Then,

(i) $\mathcal{V}_G^*$ is an output-nulling controlled-invariant subspace of (1), (2);
(ii) the set of the internal eigenvalues of $\mathcal{V}_G^*$ is the union of the set of the internal eigenvalues of $\mathcal{R}_{\mathcal{V}^*}$ and the set of the minimum-phase invariant zeros of (1), (2) and is contained in the unit disc: i.e., $\sigma((A+BF)|_{\mathcal{V}_G^*}) = \sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cup \mathcal{Z}_{MP}(A,B,C,D) \subset \mathbb{C}^{\odot}$.

Proof. Proposition (i): Equations (12) and (18) imply

$$A_F^{\prime\prime\prime} V_G^{*\prime\prime\prime} = V_G^{*\prime\prime\prime} \operatorname{diag}\{A_{11}', A_{22_S}''\}. \tag{19}$$

Hence, $\mathcal{V}_G^*$ is an $(A+BF)$-invariant subspace or, equivalently, $\mathcal{V}_G^*$ is an $(A,\mathcal{B})$-controlled invariant subspace. Moreover, (13) and (18) imply

$$C_F^{\prime\prime\prime} V_G^{*\prime\prime\prime} = O. \tag{20}$$

Hence, $\mathcal{V}_G^*$ is an output-nulling subspace. Then, (19) and (20) together imply that $\mathcal{V}_G^*$ is an output-nulling controlled-invariant subspace.

Proposition (ii) follows from (19) in light of Remark 2, Remark 5, and property (17) of the linear map F: i.e., $\sigma((A+BF)|_{\mathcal{V}_G^*}) = \sigma(A_{11}') \cup \sigma(A_{22_S}'') = \sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cup \mathcal{Z}_{MP}(A,B,C,D) \subset \mathbb{C}^{\odot}$. □

Corollary 10. Consider system (1), (2). Let the linear map F be such that $(A+BF)\,\mathcal{V}^* \subseteq \mathcal{V}^*$, $\mathcal{V}^* \subseteq \ker(C+DF)$, $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \cap \sigma((A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}) = \emptyset$, and $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}) \subset \mathbb{C}^{\odot}$. Let $\bar{T} = TT'T''$ where T, T', T'' were introduced in Lemmas 1 to 4. Let $V_G^* = \bar{T} V_G^{*\prime\prime\prime}$, where $V_G^{*\prime\prime\prime}$ was introduced in (18). Then,

$$A V_G^* - V_G^* W_G = -B L_G, \tag{21}$$

$$C V_G^* = -D L_G, \tag{22}$$

hold with $W_G = \operatorname{diag}\{A_{11}', A_{22_S}''\}$ and $L_G = F V_G^*$.

Proof. The definition of V_G^* and the invertibility of $\bar{T}$ give $V_G^{*\prime\prime\prime} = \bar{T}^{-1} V_G^*$. Hence, also in light of (12) and (13), (19) and (20) become

$$\bar{T}^{-1}(A+BF)\, V_G^* = \bar{T}^{-1} V_G^* \operatorname{diag}\{A_{11}', A_{22_S}''\}, \tag{23}$$

$$(C+DF)\, V_G^* = O. \tag{24}$$

By left-multiplying (23) with $\bar{T}$ and replacing the matrices $\operatorname{diag}\{A_{11}', A_{22_S}''\}$ and $F V_G^*$ with W_G and L_G, respectively, in both (23) and (24), one gets (21) and (22). □

6. PROPERTIES OF THE CASCADE

The synthesis of the feedforward compensator will be carried out assuming $\mathcal{V}_G^*$ as the key subspace. The system matrix and the output distribution matrix of the precompensator will be W_G and L_G, respectively. The input distribution matrix and the direct feedthrough matrix will respectively be such that the states of the feedforward compensator and the inputs of the original system be directly accessible one by one. The synthesis based on $\mathcal{V}_S^*$ would have required the use of W_S and L_S, respectively. The properties of the cascade system will then be examined. It is henceforth understood that the conditions on the linear map F specified in the previous section hold, even though they are not explicitly mentioned.

According to the previous considerations, let the feedforward compensator (3), (4) have

$$A_f = W_G, \quad B_f = [\, I_{n_f} \;\; O \,], \tag{25}$$

$$C_f = L_G, \quad D_f = [\, O \;\; I_p \,], \tag{26}$$

where $n_f = n_1 + n_S$. Hence, the cascade of the precompensator (3), (4), with (25) and (26), and the original system (1), (2) is described by

$$x_{c,t+1} = A_c\, x_{c,t} + B_c\, w_t, \tag{27}$$

$$y_t = C_c\, x_{c,t} + D_c\, w_t, \tag{28}$$

where

$$A_c = \begin{bmatrix} A & BC_f \\ O & A_f \end{bmatrix}, \quad B_c = \begin{bmatrix} BD_f \\ B_f \end{bmatrix}, \tag{29}$$

$$C_c = [\, C \;\; DC_f \,], \quad D_c = DD_f. \tag{30}$$

Lemma 11, that follows, sets the bases for the similarity transformation used to prove Theorem 12, which introduces an input-output equivalent representation of the cascade (27), (28), with (29) and (30).

Lemma 11. Consider system (27), (28), with (29) and (30). Let

$$\mathcal{J} = \operatorname{im} J = \operatorname{im} \begin{bmatrix} I_n \\ O \end{bmatrix}, \quad \mathcal{J}_c = \operatorname{im} J_c = \operatorname{im} \begin{bmatrix} V_G^* \\ I_{n_f} \end{bmatrix}. \tag{31}$$

Then,

(i) $\mathcal{J}$ is an A_c-invariant subspace;
(ii) $\mathcal{J}_c$ is an A_c-invariant subspace;
(iii) $\mathcal{J} \oplus \mathcal{J}_c = \mathcal{X}_c$, where $\oplus$ stands for the direct sum of subspaces and $\mathcal{X}_c$ denotes the state space of (27), (28).

Proof. Proposition (i) is implied by $A_c\, J = J\, A$, which holds by virtue of (29) and (31).
Proposition (ii) is implied by $A_c\, J_c = J_c\, A_f$, which holds by virtue of (29) and (31), in light of (21), (25), and (26).
Proposition (iii) follows from the comparison of the respective basis matrices J and J_c of $\mathcal{J}$ and $\mathcal{J}_c$, defined by (31). □

Theorem 12. Consider system (5), (6). Let

$$A_e = A, \quad B_e = [\, -V_G^* \;\; B \,], \tag{32}$$

$$C_e = C, \quad D_e = [\, O \;\; D \,]. \tag{33}$$

Then, system (5), (6), with (32) and (33), is an input-output equivalent representation of (27), (28), with (29) and (30): i.e., the systems (5), (6) and (27), (28) have the same reachable and observable subsystem.

Proof. Let $T_c = [J \;\; J_c]$, where J and J_c are defined by (31). Owing to Proposition (iii) of Lemma 11, T_c is invertible. Let the similarity transformation T_c be applied to (27), (28) with (29) and (30). With respect to the new coordinates, the matrices of the quadruple become

$$A_c' = T_c^{-1} A_c T_c = \begin{bmatrix} I & -V_G^* \\ O & I \end{bmatrix} \begin{bmatrix} A & BC_f \\ O & A_f \end{bmatrix} \begin{bmatrix} I & V_G^* \\ O & I \end{bmatrix}$$
$$= \begin{bmatrix} A & AV_G^* + BC_f - V_G^* A_f \\ O & A_f \end{bmatrix} = \begin{bmatrix} A & O \\ O & A_f \end{bmatrix}, \qquad (34)$$

$$B_c' = T_c^{-1} B_c = \begin{bmatrix} I & -V_G^* \\ O & I \end{bmatrix} \begin{bmatrix} BD_f \\ B_f \end{bmatrix}$$
$$= \begin{bmatrix} I & -V_G^* \\ O & I \end{bmatrix} \begin{bmatrix} O & B \\ I & O \end{bmatrix} = \begin{bmatrix} -V_G^* & B \\ I & O \end{bmatrix}, \qquad (35)$$

$$C_c' = C_c T_c = [\, C \;\; DC_f \,] \begin{bmatrix} I & V_G^* \\ O & I \end{bmatrix}$$
$$= [\, C \;\; CV_G^* + DC_f \,] = [\, C \;\; O \,], \qquad (36)$$

$$D_c' = D_c = [\, O \;\; D \,], \qquad (37)$$

where (21), (22), (25), and (26) have been taken into account. Then, one gets the quadruple (A_e, B_e, C_e, D_e), defined by (32) and (33), from the quadruple (A_c', B_c', C_c', D_c'), defined by (34)–(37), by dropping the subsystem described by the triple (A_f, I, O), which is unobservable. This implies that system (5), (6), with (32) and (33), is an input-output equivalent representation of (27), (28), with (29) and (30). □

The following statements are aimed at proving that the cascade (5), (6), with (32) and (33) satisfies Conditions $\mathcal{C}\,1$, $\mathcal{C}\,2$, and $\mathcal{C}\,3$ of the problem statement. To be more specific, a simple inspection of Definitions (32) and (33) shows that, on Assumption $\mathcal{A}\,1$, the cascade satisfies Condition $\mathcal{C}\,1$. Instead, a thorough comparison between the basic subspaces of the original system and those of the cascade is needed to prove Conditions $\mathcal{C}\,2$ and $\mathcal{C}\,3$.

Lemma 13. Consider system (1), (2) and system (5), (6), with (32) and (33). Let $\mathcal{V}^* = \max \mathcal{V}(A, B, C, D)$ and $\mathcal{V}_e^* = \max \mathcal{V}(A_e, B_e, C_e, D_e)$. Then, $\mathcal{V}_e^* = \mathcal{V}^*$.

Proof. As reviewed in Appendix A (Proposition 19), a basis matrix of $\mathcal{V}^* = \max \mathcal{V}(A, B, C, D)$ can be derived from that of $\hat{\mathcal{V}}^* = \max \mathcal{V}(\hat{A}, \hat{B}, \hat{C}, O)$, where $\hat{A}$, $\hat{B}$, and $\hat{C}$ are defined by (A.3). Similarly, a basis matrix of $\mathcal{V}_e^* = \max \mathcal{V}(A_e, B_e, C_e, D_e)$ can be derived from that of $\hat{\mathcal{V}}_e^* = \max \mathcal{V}(\hat{A}_e, \hat{B}_e, \hat{C}_e, O)$, where $\hat{A}_e$, $\hat{B}_e$, and $\hat{C}_e$ are defined according to (A.3), with A, B, C, and D replaced by A_e, B_e, C_e, and D_e, respectively. Owing to (32) and (33), $\hat{A}_e = \hat{A}$, $\hat{B}_e = \begin{bmatrix} -\hat{V}_G^* & \hat{B} \end{bmatrix}$, where $\hat{V}_G^* = \begin{bmatrix} V_G^* \\ O \end{bmatrix}$, and $\hat{C}_e = \hat{C}$. Then,

$$\hat{\mathcal{V}}_e^* = \max \mathcal{V}(\hat{A}_e, \hat{B}_e, \hat{C}_e, O) =$$
$$\max \mathcal{V}(\hat{A}, [-\hat{V}_G^* \;\; \hat{B}], \hat{C}, O) = \max \mathcal{V}(\hat{A}, \hat{B}, \hat{C}, O) = \hat{\mathcal{V}}^* \qquad (38)$$

since $\hat{\mathcal{V}}_G^* = \operatorname{im} \hat{V}_G^* \subseteq \hat{\mathcal{V}}^*$ and, for any subspace $\hat{\mathcal{H}} = \operatorname{im} \hat{H}$ contained in $\hat{\mathcal{V}}^*$, the following relation holds (Basile and Marro, 1992, Property 4.2-1): $\max \mathcal{V}(\hat{A}, [\hat{B} \;\; \hat{H}], \hat{C}, O) = \max \mathcal{V}(\hat{A}, \hat{B}, \hat{C}, O)$, Therefore, the thesis follows from (38) and, again, Proposition 19. □

Lemma 14. Consider system (1), (2) and system (5), (6), with (32) and (33). Let $\mathcal{S}^* = \min \mathcal{S}(A, B, C, D)$ and $\mathcal{S}_e^* = \min \mathcal{S}(A_e, B_e, C_e, D_e)$. Then, $\mathcal{S}_e^* = \mathcal{S}^* + \mathcal{V}_G^*$.

Proof. As reviewed in Appendix A (Proposition 20), a basis matrix of $\mathcal{S}^* = \min \mathcal{S}(A, B, C, D)$ can be derived from that of $\hat{\mathcal{S}}^* = \min \mathcal{S}(\hat{A}, \hat{B}, \hat{C}, O)$, where $\hat{A}$, $\hat{B}$, and $\hat{C}$ are defined by (A.5). Similarly, a basis matrix of $\mathcal{S}_e^* = \min \mathcal{S}(A_e, B_e, C_e, D_e)$ can be derived from that of $\hat{\mathcal{S}}_e^* = \min \mathcal{S}(\hat{A}_e, \hat{B}_e, \hat{C}_e, O)$, where $\hat{A}_e$, $\hat{B}_e$, $\hat{C}_e$ are defined according to (A.5), with A, B, C, and D respectively replaced by A_e, B_e, C_e, and D_e. Owing to (32) and (33), $\hat{A}_e$, $\hat{B}_e$, $\hat{C}_e$ have the structure

$$\hat{A}_e = \begin{bmatrix} A & -V_G^* & B \\ O & O & O \\ O & O & O \end{bmatrix}, \quad \hat{B}_e = \begin{bmatrix} O & O \\ I_{n_f} & O \\ O & I_p \end{bmatrix}, \quad \hat{C}_e = [\, C \;\; O \;\; D \,].$$

Hence, by applying Algorithm 2 to the extended triple $(\hat{A}_e, \hat{B}_e, \hat{C}_e)$ and taking into account Proposition 20, one gets

$$\hat{\mathcal{S}}_e^* = \operatorname{im} \begin{bmatrix} S^* & V_G^* & O & O \\ O & O & I_{n_f} & O \\ O & O & O & I_p \end{bmatrix} = \operatorname{im} \begin{bmatrix} S_e^* & O & O \\ O & I_{n_f} & O \\ O & O & I_p \end{bmatrix}, \qquad (39)$$

where S^* and V_G^* are basis matrices of $\mathcal{S}^*$ and $\mathcal{V}_G^*$, respectively. Hence, (39) implies $\mathcal{S}_e^* = \operatorname{im} S_e^* = \operatorname{im} [S^* \;\; V_G^*] = \mathcal{S}^* + \mathcal{V}_G^*$. □

Lemma 15. Consider system (1), (2) and system (5), (6), with (32) and (33). Let $\mathcal{R}_{\mathcal{V}_e^*} = \mathcal{V}_e^* \cap \mathcal{S}_e^*$. Then, $\mathcal{R}_{\mathcal{V}_e^*} = \mathcal{V}_G^*$.

Proof. Owing to Lemmas 13–14, and inclusions $\mathcal{R}_{\mathcal{V}^*} \subseteq \mathcal{V}_G^* \subseteq \mathcal{V}^*$, the following relations hold: $\mathcal{R}_{\mathcal{V}_e^*} = \mathcal{V}_e^* \cap \mathcal{S}_e^* = \mathcal{V}^* \cap (\mathcal{S}^* + \mathcal{V}_G^*) = \mathcal{V}^* \cap \mathcal{S}^* + \mathcal{V}_G^* = \mathcal{R}_{\mathcal{V}^*} + \mathcal{V}_G^* = \mathcal{V}_G^*$. □

Theorem 16. Consider system (1), (2) and system (5), (6), with (32) and (33). Then, system (5), (6) satisfies Condition $\mathcal{C}\,2$.

Proof. Let $\mathcal{Z}_{NMP}(A, B, C, D)$ and $\mathcal{Z}^{\circ}(A, B, C, D)$ respectively denote the set of the nonminimum-phase invariant zeros and the set of the invariant zeros with unit absolute value of (1), (2). Let $\mathcal{Z}(A_e, B_e, C_e, D_e)$ denote the set of the invariant zeros of (5), (6). Let F_e be such that $(A_e + B_e F_e)\, \mathcal{V}_e^* \subseteq \mathcal{V}_e^*$ and $\mathcal{V}_e^* \subseteq \ker (C_e + D_e F_e)$. Then,

$$\mathcal{Z}(A_e, B_e, C_e, D_e) = \sigma((A_e + B_e F_e)|_{\mathcal{V}_e^*/\mathcal{R}_{\mathcal{V}_e^*}}) \qquad (40)$$

and is independent of F_e. Let F be such that $(A + BF)\mathcal{V}^* \subseteq \mathcal{V}^*$ and $\mathcal{V}^* \subseteq \ker (C + DF)$. Then,

$$\mathcal{Z}_{NMP}(A, B, C, D) \cup \mathcal{Z}^{\circ}(A, B, C, D) = \sigma((A + BF)|_{\mathcal{V}^*/\mathcal{V}_G^*}) \qquad (41)$$

and is independent of F. Therefore, the thesis or, equivalently, the relation $\mathcal{Z}(A_e, B_e, C_e, D_e) = \mathcal{Z}_{NMP}(A, B, C, D) \cup \mathcal{Z}^{\circ}(A, B, C, D)$ follows from the comparison of (40) with (41), in light of (32), (33), Lemma 13, and Lemma 15. □

Theorem 17. Consider system (1), (2) and system (5), (6), with (32) and (33). Then, system (5), (6) satisfies Condition $\mathcal{C}\,3$.

Proof. Owing to Lemma 13, Lemma 14, and the inclusion $\mathcal{V}_G^* \subseteq \mathcal{V}^*$, the following relations hold: $\mathcal{V}_e^* + \mathcal{S}_e^* = \mathcal{V}^* + \mathcal{S}^* + \mathcal{V}_G^* = \mathcal{V}^* + \mathcal{S}^*$. □

Remark 18. (Cancellation of Invariant Zero Subsets).
A slight modification of the procedure described so far leads to a feedforward compensator achieving cancellation of any subsets of minimum-phase invariant zeros of the original system corresponding to indipendent elementary blocks in the real Jordan canonical form of matrix A'_{22}. In more detail, the similarity trasformation T'', considered in Lemma 4, must be defined in such a way that matrix A'_{22} be returned in real Jordan canonical form. Then, the definition of the key subspace $\mathcal{V}_G^*$, given in Theorem 9, must be modified so that the basis matrix $V_G^{*\prime\prime\prime}$ includes an identity matrix that selects the Jordan blocks corresponding to the sole the minimum-phase zeros to be cancelled. Still, $\mathcal{V}_G^*$ is an output-nulling controlled-invariant subspace of (1), (2) and Corollary 10 provides the linear maps to be used in the synthesis of the feedforward compensator.

7. ZERO CANCELLATION BY A CASCADED FILTER AND APPLICATION TO UNKNOWN-STATE, UNKNOWN-INPUT RECONSTRUCTION

The problem of zero cancellation by a filter connected in cascade can be stated for a system like

$$x_{t+1} = A^\top x_t + C^\top u_t, \tag{42}$$

$$y_t = B^\top x_t + D^\top u_t, \tag{43}$$

satisfying assumption

$\mathcal{A}\,1'$. $\max \mathcal{J}(A^\top, \ker B^\top) = \{0\}$.

The filter, described by

$$x_{f,t+1} = A_f^\top x_{f,t} + C_f^\top y_t, \tag{44}$$

$$\bar{y}_t = B_f^\top x_{f,t} + D_f^\top y_t, \tag{45}$$

should be such that the cascade, described by

$$x_{e,t+1} = A_e^\top x_{e,t} + C_e^\top u_t, \tag{46}$$

$$\bar{y}_t = B_e^\top x_{e,t} + D_e^\top u_t, \tag{47}$$

satisfy the conditions:

$\mathcal{C}\,1'$. $\max \mathcal{J}(A_e^\top, \ker B_e^\top) = \{0\}$;

$\mathcal{C}\,2'$. $\mathcal{Z}(A_e^\top, C_e^\top, B_e^\top, D_e^\top) = \mathcal{Z}_{NMP}(A^\top, C^\top, B^\top, D^\top) \cup \mathcal{Z}^\circ(A^\top, C^\top, B^\top, D^\top)$;

$\mathcal{C}\,3'$. $\max \mathcal{V}(A^\top, C^\top, B^\top, D^\top) \cap \min \mathcal{S}(A^\top, C^\top, B^\top, D^\top) = \{0\} \implies \max \mathcal{V}(A_e^\top, C_e^\top, B_e^\top, D_e^\top) \cap \min \mathcal{S}(A_e^\top, C_e^\top, B_e^\top, D_e^\top) = \{0\}$.

This statement of the problem is the dual counterpart of that considered in Section 2. Hence, the matrices A_f, B_f, C_f, D_f of the filter are those computed by following the procedures described in Section 4 or in Section 5.

If system (42), (43) is not only observable, but also left-invertible, then the problem of reconstructing, with an admissible delay, the unknown-input u in the presence of the unknown initial state x_0 is solvable. When the equivalent cascade system (46), (47) has been determined, the state and input reconstructor can be obtained by following the lines presented in Marro et al. (2010).

In consideration of the fact that in cascade systems without invariant zeros (i.e., cascade systems obtained from minimum-phase systems) the number of steps for reconstructing the unknown initial state (and, consequently, the number of steps for reconstructing the unknown input, which requires an additional delay of one step) depends on the number of iterations of the algorithm for computing the minimal input-containing conditioned-invariant subspace of the dual system, the solution based on the key subspace $\mathcal{V}_G^*$ is the most valuable. Indeed, with respect to the primal system $\mathcal{V}_G^* = \mathcal{V}_S^* + \mathcal{R}_{\mathcal{V}^*}$ and the image of the input distribution matrix of the cascade is either $\mathcal{B} + \mathcal{V}_G^* = \mathcal{B} + \mathcal{R}_{\mathcal{V}^*} + \mathcal{V}_S^*$ or $\mathcal{B} + \mathcal{V}_S^*$, depending on which subspace, $\mathcal{V}_G^*$ or $\mathcal{V}_S^*$ respectively, is assumed as resolving subspace. Then, owing to the fact that the image of input distribution matrix is the first step of the algorithm for costructing the minimal input-containing conditioned-invariant subspace and in light of Lemma 14, the number of iterations for computing the minimal input-containing conditioned-invariant subspace of the cascade defined assuming $\mathcal{V}_G^*$ as key subspace is not greater than that for computing the corresponding subspace when $\mathcal{V}_S^*$ is picked as the crucial subspace.

8. AN ILLUSTRATIVE EXAMPLE

In this section, a numerical example is worked out in order to describe the practical impact of the theoretical aspects discussed up to this point. First, the synthesis of two different feedforward compensators is presented, according to the methods discussed in Sections 4 and 5. Then, their respective application to the solution of the unknown-state, unknown-input reconstruction problem formulated for the dual counterpart of the original system is outlined, in accordance with Section 7. The computational support consists of the Matlab files implementing the geometric approach algorithms first appeared in Basile and Marro (1992) and now available online in an upgraded version. The variables are displayed in scaled fixed point format with five digits, although computations are made in floating point precision. Consider a system like (1), (2), obtained as the dual counterpart of the sampled-data system derived with ZOH-discretization method and sampling time $T_s = 0.5\,\mathrm{s}$ from the nominal, continuous-time, linear model of an aircraft considered in Bokor and Balas (2004): i.e.,

$$A = \begin{bmatrix} 0.1724 & 0.5659 & 0 & 0 & 0 & 0 \\ -0.5659 & 0.1724 & 0 & 0 & 0 & 0 \\ 0.4460 & 0.4454 & 0.0000 & 0 & -0.0000 & 0 \\ 1.0249 & 0.7843 & 0.0000 & 0.0000 & -0.0001 & 0 \\ -3.7470 & -4.0004 & 0 & 0 & 0.0000 & 0 \\ -0.0030 & -0.0013 & 0 & 0 & 0 & 0.9512 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.0100 & -0.4800 & 0.0300 & 0.2600 \\ 0.0900 & -0.5900 & 0.0900 & -0.0700 \\ 0.0700 & 0 & -0.0600 & 0.0100 \\ 0 & 0 & 0 & 0 \\ 0 & -49.5100 & 0 & 0 \\ 0 & -0.0026 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} -2.9312 & -0.6172 & 0 & 0 & 0.0495 & 0 \\ 1.2475 & 0.5529 & 0 & 0 & -0.0010 & 0 \\ -2.5117 & 2.5789 & 0 & 0 & 0.0277 & 0 \end{bmatrix},$$

$D = O_{3\times 4}$.

The system considered satisfies Assumption $\mathcal{A}\,1$, since $\mathcal{R}=\mathbb{R}^6$. The basic subspaces are

$$\mathcal{V}^* = \mathrm{im}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$\mathcal{S}^* = \mathrm{im}\begin{bmatrix} 0.0097 & -0.9649 & -0.2386 & -0.1092 & -0.0000 \\ 0.0119 & 0.2599 & -0.8098 & -0.5259 & -0.0000 \\ 0 & -0.0371 & 0.5359 & -0.8435 & -0.0000 \\ 0 & 0 & 0 & 0 & 1.0000 \\ 0.9999 & 0.0063 & 0.0120 & 0.0073 & -0.0000 \\ 0.0001 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{bmatrix}.$$

The system is right-invertible, since $\mathcal{V}^*+\mathcal{S}^*=\mathcal{X}=\mathbb{R}^6$. The system is non-left-invertible, since

$$\mathcal{R}_{\mathcal{V}^*} = \mathcal{V}^* \cap \mathcal{S}^* = \mathrm{im}\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

Hence, the number of the invariant zeros is $\dim \mathcal{V}^* - \dim \mathcal{R}_{\mathcal{V}^*} = 1$, while the number of the internal assignable eigenvalues of $\mathcal{V}^*$ is $\dim \mathcal{R}_{\mathcal{V}^*} = 2$. The invariant zero is $z=0.9512$. Hence, $z \subset \mathbb{C}^{\odot}$ and can be cancelled by a feedforward compensator designed according to the procedures described in Sections 4 and 5. Since $\mathcal{R}_{\mathcal{V}^*} \neq \{0\}$, the two procedures give different precompensators. In particular, according to the further condition on F specified in Section 5, the set $\sigma((A+BF)|_{\mathcal{R}_{\mathcal{V}^*}})$ can be assigned, e.g., equal to $\{0.4,\, 0.6\}$. According to the procedure described in Section 4, the resolving subspace is represented by

$$\mathcal{V}_S^* = \mathrm{im}\, V_S^* = \mathrm{im}\,[\,0\ 0\ 0\ 0\ 0\ 1\,]^\top,$$

with respect to the original coordinates. The matrices W_S and L_S are

$$W_S = 0.9512, \quad L_S = 10^{-9}\begin{bmatrix} -0.1407 \\ 0.0000 \\ 0.1253 \\ -0.0199 \end{bmatrix},$$

and the feedforward compensator is accordingly defined by $A_f = W_S$, $B_f = [1\ 0_{1\times 4}]$, $C_f = L_S$, and $D_f = [O_{4\times 1}\ I_4]$. Instead, following the procedure described in Section 5, one gets the resolving subspace represented by

$$\mathcal{V}_G^* = \mathrm{im}\, V_G^* = \mathrm{im}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with respect to the original coordinates. The matrices W_G and L_G are 3×3 and 4×3 matrices solving (21) and (22). The feedforward compensator is defined accordingly by $A_f = W_G$, $B_f = [I_3\ O_{3\times 4}]$, $C_f = L_G$, and $D_f = [O_{4\times 3}\ I_4]$. As one can easily check, the cascade, defined according to (32) and (33), satisfies Conditions $\mathcal{C}\,1$–$\mathcal{C}\,3$. The filters obtained as the dual counterparts of the feedforward compensators devised above can be used in the solution of the unknown-state, unknown-input reconstruction problem formulated for the dual counterpart of the given system, as described in Section 7. Figures 1 and 2 show the results of a simulation run with the initial state, considered unknown, $x_0 = [1\ 2\ 3\ 4\ 5\ 6]^\top$ and the input signals, also supposed unknown, respectively given by a step, a sinusoid, and a sawtooth (upper plots in both figures). Figure 1 refers to the use of the filter designed on the basis of $\mathcal{V}_S^*$, while Fig. 2 concerns the employment of the filter whose synthesis exploits $\mathcal{V}_G^*$. As explained in Section 7, the use of the filter with the greater dynamic order leads to a faster reconstruction because it corresponds to a greater numbers of additional outputs in the cascade, hence, to more information available on the system.

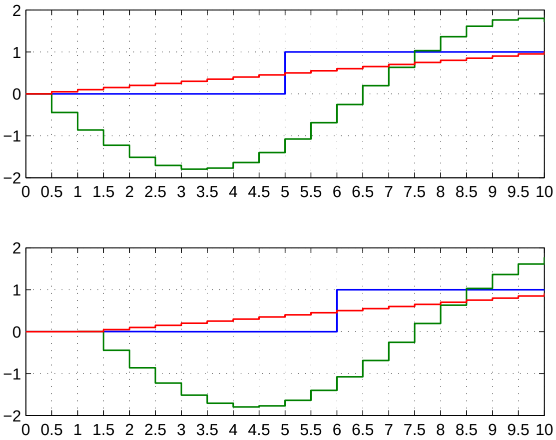

Fig. 1. Reconstruction with the minimal-order filter: unknown input (upper plot) and reconstructed input (lower plot)

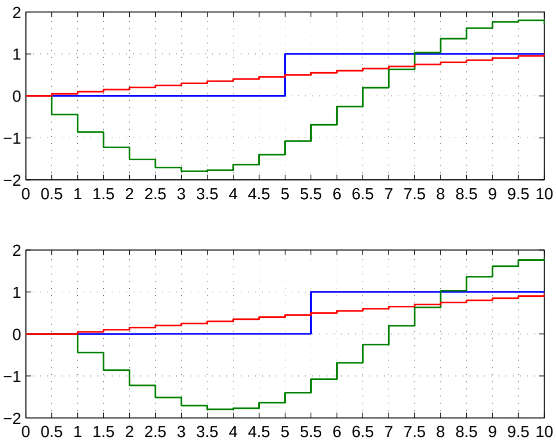

Fig. 2. Reconstruction with the minimum number of steps: unknown input (upper plot) and reconstructed input (lower plot)

9. CONCLUSION

Two geometric methods for zero cancellation in discrete-time linear multivariable systems were compared and their impact on reconstruction of unknown states and inputs was analyzed. The most convenient solution in terms of minimal delay in state and input reconstruction was pointed out. A numerical example was discussed.

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Appendix A. GEOMETRIC APPROACH BACKGROUND

This appendix reviews some definitions and properties of the geometric approach, used in this work. The reader is also referred to Wonham (1985) and Basile and Marro (1992). Refer to system (1), (2). Subspaces playing a crucial role are $\mathcal{B}$, the image of B, $\mathcal{C}$, the kernel of C, $\mathcal{R}=\min\mathcal{J}(A,\mathcal{B})$, the reachable subspace of (A,B) or, equivalently, the minimal A-invariant subspace containing $\mathcal{B}$, $\mathcal{Q}=\max\mathcal{J}(A,\mathcal{C})$, the unobservable subspace of (A,C) or, equivalently, the maximal A-invariant subspace contained in $\mathcal{C}$, $\mathcal{V}^*=\max\mathcal{V}(A,B,C,D)$, the maximal output-nulling controlled-invariant subspace of (1), (2), $\mathcal{S}^*=\min\mathcal{S}(A,B,C,D)$, the minimal input-containing conditioned-invariant subspace of (1), (2), $\mathcal{R}_{\mathcal{V}^*}=\mathcal{V}^*\cap\mathcal{S}^*$, the reachability subspace on $\mathcal{V}^*$. These subspaces are further specified with small subscripts when the same subscripts are used for the systems they refer to. Some properties of the abovementioned subspaces, functional to the developments of this work, are reviewed below. A subspace $\mathcal{V}\subseteq\mathcal{X}$ is an output-nulling controlled-invariant subspace of (1), (2) if and only if a linear map F exists, such that $(A+BF)\,\mathcal{V}\subseteq\mathcal{V}$ and $\mathcal{V}\subseteq\ker(C+DF)$. A subspace $\mathcal{S}\subseteq\mathcal{X}$ is an input-containing conditioned-invariant subspace of (1), (2) if and only if a linear map G exists, such that $(A+GC)\,\mathcal{S}\subseteq\mathcal{S}$ and $\mathcal{S}\supseteq\operatorname{im}(B+GD)$. Let the linear map F be such that $(A+BF)\,\mathcal{V}^*\subseteq\mathcal{V}^*$ and $\mathcal{V}^*\subseteq\ker(C+DF)$, then $(A+BF)\,\mathcal{R}_{\mathcal{V}^*}\subseteq\mathcal{R}_{\mathcal{V}^*}$ and $\mathcal{R}_{\mathcal{V}^*}\subseteq\ker(C+DF)$ hold with the same F. The spectrum of $(A+BF)|_{\mathcal{R}_{\mathcal{V}^*}}$ is assignable. The spectrum of $(A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}$ is fixed. The spetrum of $(A+BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}$ is also known as the set of the internal unassignable eigenvalues of $\mathcal{V}^*$ or, equivalently, as the set $\mathcal{Z}(A,B,C,D)$ of the invariant zeros of (1), (2). With a slight abuse of nomenclature, the invariant zeros of (1), (2) contained in $\mathbb{C}^{\odot}$ are also called minimum-phase invariant zeros and their set is denoted by $\mathcal{Z}_{MP}(A,B,C,D)$. Similarly, the invariant zeros contained in $\mathbb{C}^{\otimes}$ are named nonminimum-phase invariant zeros and their set is denoted by $\mathcal{Z}_{NMP}(A,B,C,D)$. Moreover, the set of the invariant zeros on $\mathbb{C}^{\circ}$ is denoted by $\mathcal{Z}^{\circ}(A,B,C,D)$. Further crucial notions are those of right-invertibility and left-invertibility of a system. A geometric condition equivalent to the property of system (1), (2) of being right-invertible is $\mathcal{V}^*+\mathcal{S}^*=\mathcal{X}$. A geometric condition equivalent to the property of system (1), (2) of being left-invertible is $\mathcal{R}_{\mathcal{V}^*}=\mathcal{V}^*\cap\mathcal{S}^*=\{0\}$. Some geometric properties of systems with direct feedthrough terms can be efficiently studied through suitably defined systems without feedthrough terms. This approach is used in this work, where the maximal output-nulling controlled-invariant subspace and the minimal input-containing conditioned-invariant subspace are respectively obtained through the algorithms defined for triples like

$$\hat{x}_{t+1}=\hat{A}\,\hat{x}_t+\hat{B}\,u_t, \tag{A.1}$$

$$y_t=\hat{C}\,\hat{x}_t. \tag{A.2}$$

Algorithm 1. Consider system (A.1), (A.2). The subspace $\hat{\mathcal{V}}^*=\max\mathcal{V}(\hat{A},\hat{B},\hat{C},O)$ is the last term of the sequence $\hat{\mathcal{V}}_0=\hat{\mathcal{C}}$, $\hat{\mathcal{V}}_i=\hat{A}^{-1}(\hat{\mathcal{V}}_{i-1}+\hat{\mathcal{B}})\cap\hat{\mathcal{C}}$, $i=1,2,\ldots,k$, where k is the least integer such that $\hat{\mathcal{V}}_{k+1}=\hat{\mathcal{V}}_k$.

Algorithm 2. Consider system (A.1), (A.2). The subspace $\hat{\mathcal{S}}^*=\min\mathcal{S}(\hat{A},\hat{B},\hat{C},O)$ is the last term of the sequence $\hat{\mathcal{S}}_0=\hat{\mathcal{B}}$, $\hat{\mathcal{S}}_i=\hat{A}(\hat{\mathcal{S}}_{i-1}\cap\hat{\mathcal{C}})+\hat{\mathcal{B}}$, $i=1,2,\ldots,k$, where k is the least integer such that $\hat{\mathcal{S}}_{k+1}=\hat{\mathcal{S}}_k$.

The following propositions relate the basic subspaces of a quadruple, like (1), (2), to the corresponding subspaces of triples, like (A.1), (A.2), conveniently constructed. Proofs are omitted for the sake of brevity.

Proposition 19. Consider system (1), (2). Let $\mathcal{V}^*=\max\mathcal{V}(A,B,C,D)$. Consider system (A.1), (A.2), with

$$\hat{A}=\begin{bmatrix} A & O \\ C & O \end{bmatrix},\quad \hat{B}=\begin{bmatrix} B \\ D \end{bmatrix},\quad \hat{C}=[\,O\ \ I_q\,]. \tag{A.3}$$

Let V^* be a basis matrix of $\mathcal{V}^*$. Then,

$$\hat{\mathcal{V}}^*=\operatorname{im}\hat{V}^*=\operatorname{im}\begin{bmatrix} V^* \\ O \end{bmatrix}. \tag{A.4}$$

Proposition 20. Consider system (1), (2). Let $\mathcal{S}^*=\min\mathcal{S}(A,B,C,D)$. Consider system (A.1), (A.2), with

$$\hat{A}=\begin{bmatrix} A & B \\ O & O \end{bmatrix},\quad \hat{B}=\begin{bmatrix} O \\ I_p \end{bmatrix},\quad \hat{C}=[\,C\ \ D\,]. \tag{A.5}$$

Let S^* be a basis matrix of $\mathcal{S}^*$. Then,

$$\hat{\mathcal{S}}^*=\operatorname{im}\hat{S}^*=\operatorname{im}\begin{bmatrix} S^* & O \\ O & I_p \end{bmatrix}. \tag{A.6}$$