Curtin University of Technology

Invariant Subspaces: Controllability and Observability

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Geometric Control Theory for Linear Systems

Block 1: Foundations [10:30 - 12.30]:

- Talk 1: *Motivation and historical perspective*, **G. Marro** [10:30 - 11:00]
- Talk 2: *Invariant subspaces*, **L. Ntogramatzidis** [11:00 - 11:30]
- Talk 3: *Controlled invariance and invariant zeros*, **D. Prattichizzo** [11:30 - 12:00]
- Talk 4: *Conditioned invariance and state observation*, **F. Morbidi** [12:00 - 12:30]

Block 2: Problems and applications [15:30 - 17.30]:

- Talk 5: *Stabilization and self-bounded subspaces*, **L. Ntogramatzidis** [15:30 - 16:00]
- Talk 6: *Disturbance decoupling problems*, **L. Ntogramatzidis** [16:00 - 16:30]
- Talk 7: *LQR and H_2 control problems*, **D. Prattichizzo** [16:30 - 17:00]
- Talk 8: *Spectral factorization and H_2-model following*, **F. Morbidi** [17:00 - 17:30]

Outline

- **Basic operations on subspaces**
- A-invariant subspaces
- The most important solved problems

State-Space Models

Consider the LTI system Σ ruled by

$$\Sigma : \quad \begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) & x(0) = x_0 \\ y(t) = C\,x(t) + D\,u(t) \end{cases}$$

where, for all $t \geq 0$,

- $x(t) \in \mathbb{R}^n$ is the state
- $u(t) \in \mathbb{R}^m$ is the control input
- $y(t) \in \mathbb{R}^p$ is the output.

Subspaces

A subspace $\mathcal{X}$ of $\mathbb{R}^n$ can be represented by a **basis matrix** X, i.e., a matrix whose columns are linearly independent and span $\mathcal{X}$.

Subspaces

A subspace $\mathcal{X}$ of $\mathbb{R}^n$ can be represented by a **basis matrix** X, i.e., a matrix whose columns are linearly independent and span $\mathcal{X}$.

Hence, a basis matrix X of $\mathcal{X}$ is such that

$$\boxed{\begin{aligned} \operatorname{im} X &= \mathcal{X} \\ \ker X &= \{0\} \end{aligned}}$$

For computational robustness, we require the columns of X to be an orthonormal basis of $\mathcal{X}$.

Basic Operations

Matrix $A \in \mathbb{R}^{m \times n}$ is identified with the corresponding linear map between $\mathbb{R}^n$ and $\mathbb{R}^m$.

$\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$: subspaces of $\mathbb{R}^n$.

$\mathcal{H}$: subspace of $\mathbb{R}^m$.

Basic operations:

- *sum:* $\mathcal{X} + \mathcal{Y} = \{x + y \in \mathbb{R}^n \mid x \in \mathcal{X} \ \text{ and } \ y \in \mathcal{Y}\}$
- *linear transformation:* $A\,\mathcal{X} = \{w \in \mathbb{R}^m \mid \exists\, x \in \mathcal{X} \,:\, A\,x = w\}$
- *orthogonal complementation:* $\mathcal{X}^\perp = \{y \in \mathbb{R}^n \mid y^\top x = 0 \ \forall\, x \in \mathcal{X}\}$
- *intersection:* $\mathcal{X} \cap \mathcal{Y} = \{z \in \mathbb{R}^n \mid z \in \mathcal{X} \ \text{ and } \ z \in \mathcal{Y}\}$
- *inverse linear transformation:* $A^{-1}\,\mathcal{H} = \{x \in \mathbb{R}^n \mid A\,x \in \mathcal{H}\}$
  (Notice: matrix A needs not be invetible)

Basic Relations

$$
\begin{aligned}
\mathcal{X} \cap (\mathcal{Y} + \mathcal{Z}) &\supseteq (\mathcal{X} \cap \mathcal{Y}) + (\mathcal{X} \cap \mathcal{Z}) \\
\mathcal{X} + (\mathcal{Y} \cap \mathcal{Z}) &\subseteq (\mathcal{X} + \mathcal{Y}) \cap (\mathcal{X} + \mathcal{Z}) \\
(\mathcal{X}^\perp)^\perp &= \mathcal{X} \\
(\mathcal{X} + \mathcal{Y})^\perp &= \mathcal{X}^\perp \cap \mathcal{Y}^\perp \\
(\mathcal{X} \cap \mathcal{Y})^\perp &= \mathcal{X}^\perp + \mathcal{Y}^\perp \\
A\,(\mathcal{X} \cap \mathcal{Y}) &\subseteq A\,\mathcal{X} \cap A\,\mathcal{Y} \\
A\,(\mathcal{X} + \mathcal{Y}) &= A\,\mathcal{X} + A\,\mathcal{Y} \\
A^{-1}\,(\mathcal{H}_1 \cap \mathcal{H}_2) &= A^{-1}\,\mathcal{H}_1 \cap A^{-1}\,\mathcal{H}_2 \\
A^{-1}\,(\mathcal{H}_1 + \mathcal{H}_2) &\supseteq A^{-1}\,\mathcal{H}_1 + A^{-1}\,\mathcal{H}_2 \\
(A^{-1}\,\mathcal{H})^\perp &= A^\top \mathcal{H}^\perp \\
A\,\mathcal{X} \subseteq \mathcal{H} &\Leftrightarrow A^\top \mathcal{H}^\perp \subseteq \mathcal{X}^\perp
\end{aligned}
$$

Grassman Manifold

For every pair of subspaces $\mathcal{X}_1, \mathcal{X}_2$ of $\mathcal{X}$,

1. $\mathcal{X}_1 + \mathcal{X}_2$ is the smallest subspace of $\mathcal{X}$ containing both $\mathcal{X}_1$ and $\mathcal{X}_2$
2. $\mathcal{X}_1 \cap \mathcal{X}_2$ is the largest subspace of $\mathcal{X}$ contained in both $\mathcal{X}_1$ and $\mathcal{X}_2$

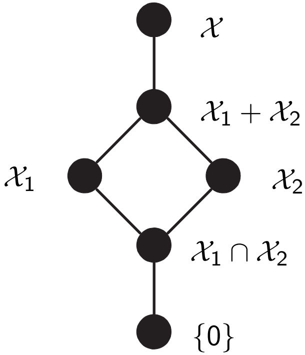


Image in MATLAB®

The overall numerical robustness is held up by the routine `ima.m`, based on the QR decomposition.

`>> Q=ima(X,[flag])`: routine `ima.m` performs the orthonormalization of a set of vectors given as the colums of the matrix X and returns them as the columns of matrix Q.

- $\mathrm{flag} = 0$: re-ordering of vectors during the orthonormalization process is **not** allowed
- Re-ordering is allowed otherwise

Example:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow \mathrm{ima}(A) = \begin{bmatrix} -0.8165 & 0 \\ -0.4082 & -0.7071 \\ 0 & 0 \\ -0.4082 & 0.7071 \end{bmatrix}, \ \mathrm{ima}(A,0) = \begin{bmatrix} 0.7071 & -0.4082 \\ 0 & -0.8165 \\ 0 & 0 \\ 0.7071 & 0.4082 \end{bmatrix}$$

Orthogonal Complement in MATLAB®

$$\mathcal{X}^{\perp} = \{y \in \mathbb{R}^n \mid y^\top x = 0 \ \forall x \in \mathcal{X}\}$$

`>> Q=ortco(X)`: routine `ortco.m` computes the orthogonal complement of $\operatorname{im} X$ as

```
[m,n]=size(ima(X,0));
M=ima([X,eye(n)],0); Q=M(:,n+1:m);
```

Example:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \to \quad \operatorname{ortco}(A) = \begin{bmatrix} 0.5774 & 0 \\ -0.5774 & 0 \\ 0 & 1 \\ -0.5774 & 0 \end{bmatrix}$$

Sum in MATLAB®

$$\mathcal{X}+\mathcal{Y} = \{x+y \in \mathbb{R}^n \mid x \in \mathcal{X} \quad \text{and} \quad y \in \mathcal{Y}\}$$

`>> Q=sums(X,Y)` routine `sums.m` computes the sum of $\operatorname{im} A$ and $\operatorname{im} B$ as

Q=ima([X,Y]);

Example:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \rightarrow \quad \mathrm{sums}(A,B) = \begin{bmatrix} 0.7071 & 0.4082 & 0.5774 \\ 0 & 0.8165 & -0.5774 \\ 0 & 0 & 0 \\ 0.7071 & -0.4082 & -0.5774 \end{bmatrix}$$

Intersection in MATLAB®

$$\mathcal{X} \cap \mathcal{Y} = \{z \in \mathbb{R}^n \,|\, z \in \mathcal{X} \quad \text{and} \quad z \in \mathcal{Y}\}$$

We exploit

$$(\mathcal{X} + \mathcal{Y})^\perp = \mathcal{X}^\perp \cap \mathcal{Y}^\perp \qquad \Longrightarrow \qquad \mathcal{X} \cap \mathcal{Y} = (\mathcal{X}^\perp + \mathcal{Y}^\perp)^\perp$$

`>> Q=ints(X,Y)` routine `ints.m` computes the intersection of $\operatorname{im} X$ and $\operatorname{im} Y$ as

Q=ortco(sums(ortco(X),ortco(Y)));

Example:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad \rightarrow \quad \text{ints}(A, B) = \begin{bmatrix} 0 \\ -0.7071 \\ 0 \\ 0.7071 \end{bmatrix}$$

Inverse Map in MATLAB®

$$A^{-1}\mathcal{H} = \{x \in \mathbb{R}^n \,|\, Ax \in \mathcal{H}\}$$

`>> Q=invt(A,H)` routine `invt.m` computes the subspace $A^{-1}\mathcal{H}$, inverse transform of $\mathcal{H} = \mathrm{im} H$ with respect to the linear map A. It exploits

$$A^{-1}\mathcal{H} = (A^{\mathsf{T}}\mathcal{H}^{\perp})^{\perp}$$

```
Q=ortco(A'*ortco(H));
```

Example:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 4 \\ 3 \\ 0 \\ 1 \end{bmatrix} \quad \to \quad \mathrm{invt}(A,H) = \begin{bmatrix} 0.5976 & 0 \\ 0.3586 & 0.8944 \\ -0.7171 & 0.4472 \end{bmatrix}$$

Null-space in MATLAB®

$$\ker A = \{x \in \mathbb{R}^n \mid A\,x = 0\}$$

`>> Q=ker(A)` routine `ker.m` computes the null-space of matrix A through

$$\ker A = (\mathrm{im}\, A^\top)^\perp$$

Q = ortco(A');

Example:

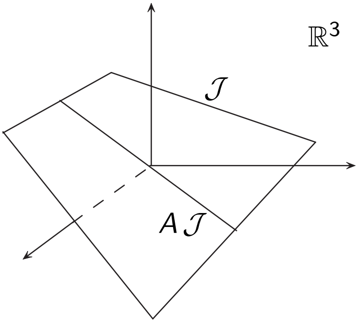
$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \to \quad \ker(A) = \begin{bmatrix} -0.5774 \\ -0.5774 \\ 0.5774 \end{bmatrix}$$

Invariant Subspaces

Given $A \in \mathbb{R}^{m \times n}$, a subspace $\mathcal{J}$ of $\mathbb{R}^n$ is an A-invariant if

$$A\mathcal{J} \subseteq \mathcal{J}$$

i.e., $Ax \in \mathcal{J}$ for all $x \in \mathcal{J}$.

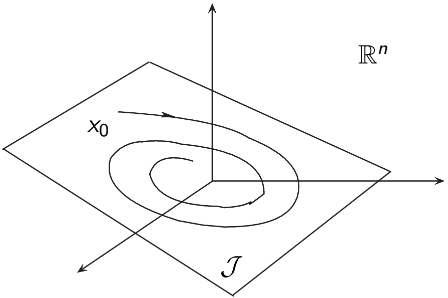


Properties of Invariant Subspaces

Let J be a basis matrix of $\mathcal{J}$; the following are equivalent:

1. $\mathcal{J}$ is A-invariant;
2. $\mathcal{J}$ is a locus of state trajectories of

$$\dot{x}(t) = A\,x(t), \qquad x(0) = x_0 \in \mathcal{J}$$



Decomposition of Invariant Subspaces

Let J be a basis matrix of the A-invariant $\mathcal{J}$. Let $T=[J\ \ T_c]$, with T_c such that T is non-singular. Then

$$A' = T^{-1} A\, T = \begin{bmatrix} A'_{11} & A'_{12} \\ O & A'_{22} \end{bmatrix}$$

The system written w.r.t. these coordinates is

$$\begin{aligned} \dot{x}_1(t) &= A'_{11}\, x_1(t) + A'_{12}\, x_2(t) & \qquad x_1(0) &= x_{10} \\ \dot{x}_2(t) &= A'_{22}\, x_2(t) & \qquad x_2(0) &= x_{20} \end{aligned}$$

The MATLAB® routine

```
>> [P,Q]=stabi(A,J)
```

computes the matrices $P = A'_{11}$ of $Q = A'_{22}$.

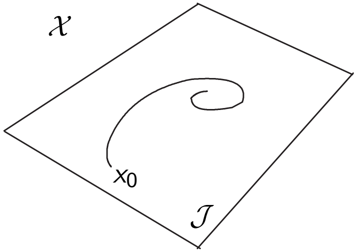
Internal Components

$$\dot{x}_1(t) = A'_{11}\, x_1(t) + A'_{12}\, x_2(t) \qquad x_1(0) = x_{10}$$
$$\dot{x}_2(t) = A'_{22}\, x_2(t) \qquad x_2(0) = x_{20}$$

If $x_{20} = 0$ $\;(x(0) \in \mathcal{J})$, then $x_2(t) = 0$ $\forall t$: the motion on $\mathcal{J}$ is described only by A'_{11}:

$$\dot{x}_1(t) = A'_{11}\, x_1(t), \quad x_1(0) = x_{10}$$

If A'_{11} is stable, $x(t) \stackrel{t\to\infty}{\longrightarrow} 0$:

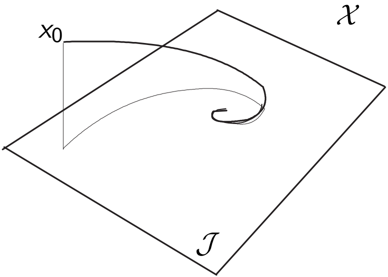


External Components

$$\dot{x}_1(t) = A'_{11}\, x_1(t) + A'_{12}\, x_2(t) \qquad x_1(0) = x_{10}$$
$$\dot{x}_2(t) = A'_{22}\, x_2(t) \qquad x_2(0) = x_{20}$$

If $x_{20} \neq 0$ $(x(0) \notin \mathcal{J})$, the state trajectory converges to $\mathcal{J}$ if and only if submatrix A_{22} is stable:

$$\dot{x}_2(t) = A'_{22}\, x_2(t), \qquad x_2(0) = x_{20} \neq 0$$



Complementability

An A-invariant subspace $\mathcal{J} \subseteq \mathcal{X}$ is *complementable* if an A-invariant subspace $\mathcal{J}_c$ exists such that

$$\mathcal{J} \oplus \mathcal{J}_c = \mathcal{X}$$

and $\mathcal{J}_c$ is a *complement* of $\mathcal{J}$.

Consider again the change of basis $T = [J \;\; T_c]$. Subspace $\mathcal{J}$ is complementable if and only if the *Sylvester equation*

$$A'_{11}\, X - X\, A'_{22} = -A'_{12}$$

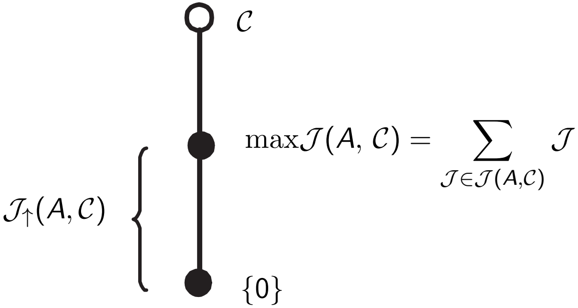
admits a solution. If so, a basis of $\mathcal{J}_c$ is $J_c = J\, X + T_2$. A sufficient condition is that matrices A'_{11} and A'_{22} have no common eigenvalues.

A-invariants contained in a given subspace

Let $\mathcal{C}$ be a subspace of $\mathcal{X}$. The set of all the A-invariant subspaces contained in $\mathcal{C}$ is

$$\mathcal{J}(A,\mathcal{C}) = \{\mathcal{J} \text{ subspace of } \mathcal{X} \mid A\,\mathcal{J} \subseteq \mathcal{J} \text{ and } \mathcal{J} \subseteq \mathcal{C}\}$$

The set $\mathcal{J}(A,\mathcal{C})$ is closed under addition, hence it admits a maximum:



Computation of $\max \mathcal{J}(A, \mathcal{C})$

Consider the sequence

$$\begin{aligned} \mathcal{Z}_1 &= \mathcal{C} \\ \mathcal{Z}_i &= \mathcal{C} \cap A^{-1}\, \mathcal{Z}_{i-1} \quad i = 2, 3, \ldots \end{aligned}$$

$\max \mathcal{J}(A, \mathcal{C})$ is obtained when the sequence stops (i.e., when $\mathcal{Z}_{i+1} = \mathcal{Z}_i$).

The MATLAB® routine

```
>> Q=maxinv(A,C)
```

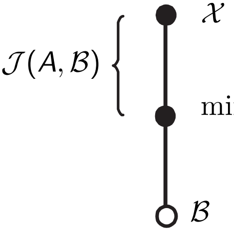
computes a basis matrix Q of $\max \mathcal{J}(A, \mathrm{im}\, C)$.

A-invariants containing a given subspace

Let $\mathcal{B}$ be a subspace of $\mathcal{X}$. The set of all the A-invariant subspaces containing $\mathcal{B}$ is

$$\mathcal{J}(A,\mathcal{B}) = \{\mathcal{J} \text{ subspace of } \mathcal{X} \mid A\,\mathcal{J} \subseteq \mathcal{J} \text{ and } \mathcal{J} \supseteq \mathcal{B}\}$$

$\mathcal{J}(A,\mathcal{B})$ is closed under intersection, hence it admits a minimum:



$$\min \mathcal{J}(A,\,\mathcal{B}) = \bigcap_{\mathcal{J}\in\mathcal{J}(A,\mathcal{B})} \mathcal{J}$$

Computation of $\min \mathcal{J}(A, \mathcal{B})$

Consider the sequence

$$\begin{aligned} \mathcal{Z}_1 &= \mathcal{B} \\ \mathcal{Z}_i &= \mathcal{B} + A\,\mathcal{Z}_{i-1} \quad i = 2, 3, \ldots \end{aligned}$$

$\min \mathcal{J}(\mathrm{A}, \mathcal{B})$ is obtained when the sequence stops (i.e., when $\mathcal{Z}_{i+1} = \mathcal{Z}_i$).

The MATLAB® routine

```
>> Q=mininv(A,B)
```

computes a basis matrix Q of $\min \mathcal{J}(A, \mathrm{im} B)$.

Duality

The following dualities hold

$$\begin{aligned} \max \mathcal{J}(A,\mathcal{C}) &= \left(\min \mathcal{J}(A^{\mathsf{T}},\mathcal{C}^{\perp})\right)^{\perp} \\ \min \mathcal{J}(A,\mathcal{B}) &= \left(\max \mathcal{J}(A^{\mathsf{T}},\mathcal{B}^{\perp})\right)^{\perp} \end{aligned}$$

Controllability

Consider

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t)$$

The reachable subspace of (A, B) is the set of all the states that can be reached from the origin in any finite time, is the minimum A-invariant containing $\mathrm{im}\, B$, i.e.,

$$\boxed{\mathcal{R} \;=\; \min \mathcal{J}(A, \mathrm{im}\, B)}$$

- If $\mathcal{R} = \mathcal{X}$, the pair (A, B) is said to be *completely controllable*.
- If $\mathcal{R} \neq \mathcal{X}$ but $\mathcal{R}$ is externally stable, the pair (A, B) is said to be *stabilizable*.

Observability

Consider

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$
$$y(t) = C\,x(t)$$

The unobservable subspace of (A, C) is the set of all the initial states that cannot be recognized from the output function in any finite time interval, is the maximum A-invariant contained in $\ker\, C$, i.e.,

$$\mathcal{Q} \;=\; \max \mathcal{J}(A, \ker C)$$

- If $\mathcal{Q} = \{0\}$, the pair (A, C) is said to be *completely observable*.
- If $\mathcal{Q} \neq \{0\}$ but $\mathcal{Q}$ is internally stable, the pair (A, C) is said to be *detectable*.