Controlled invariance and invariant zeros

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Block 1: Foundations [10:30 - 12.30]:

- Talk 1: *Motivation and historical perspective*, **G. Marro** [10:30 - 11:00]
- Talk 2: *Invariant subspaces*, **L. Ntogramatzidis** [11:00 - 11:30]
- Talk 3: *Controlled invariance and invariant zeros*, **D. Prattichizzo** [11:30 - 12:00]
- Talk 4: *Conditioned invariance and state observation*, **F. Morbidi** [12:00 - 12:30]

Block 2: Problems and applications [15:30 - 17.30]:

- Talk 5: *Stabilization and self-bounded subspaces*, **L. Ntogramatzidis** [15:30 - 16:00]
- Talk 6: *Disturbance decoupling problems*, **L. Ntogramatzidis** [16:00 - 16:30]
- Talk 7: *LQR and H_2 control problems*, **D. Prattichizzo** [16:30 - 17:00]
- Talk 8: *Spectral factorization and H_2-model following*, **F. Morbidi** [17:00 - 17:30]

Outline

- **Controlled invariance**
- Internal and external eigenvalues
- Invariant zeros
- Disturbance decoupling problem

Controlled Invariant Subspaces

Definition

Given a linear map $A : \mathcal{X} \to \mathcal{X}$ and a subspace $\mathcal{B} \subseteq \mathcal{X}$, a subspace $\mathcal{V} \subseteq \mathcal{X}$ is an $(A, \mathcal{B})$-*controlled invariant* if

$$\boxed{A\mathcal{V} \subseteq \mathcal{V} + \mathcal{B}}$$

Let B be a basis matrix of $\mathcal{B}$: the following statements are equivalent:

- $\mathcal{V}$ is an $(A, \mathcal{B})$-controlled invariant
- a matrix F exists such that $(A + BF)\mathcal{V} \subseteq \mathcal{V}$
- matrices X and U exist such that $A\,V = V\,X + B\,U$
- $\mathcal{V}$ is a locus of trajectories of the pair (A, B)

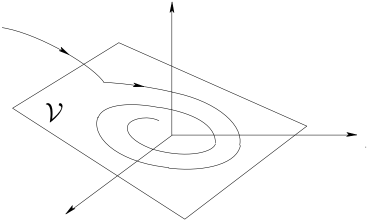


Fig: The controlled invariant as a locus of trajectories.

The maximum controlled invariant contained in $\mathcal{C}$

The sum of two controlled invariants is a controlled invariant. The intersection is not. Let $\mathcal{V}(A, \mathcal{B}, \mathcal{C})$ be the set of all the $(A, \mathcal{B})$-controlled invariants contained in the subspace $\mathcal{C} \subseteq \mathcal{X}$.

Property

$(\mathcal{V}(A,\ \mathcal{B},\ \mathcal{C}), +; \subseteq)$ is a non-distributive modular upper semilattice.

$\mathcal{V}^*_{(\mathcal{B},\mathcal{E})} = \max \mathcal{V}(A,\ \mathcal{B},\ \mathcal{C})$ is the maximal $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{C}$.

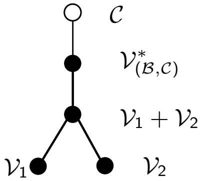


Fig: The upper semilattice $(\mathcal{V}(A,\ \mathcal{B},\ \mathcal{C}), +; \subseteq)$

Algorithm (Computation of $\mathcal{V}^*_{(\mathcal{B},\mathcal{C})}$)

$$\mathcal{V}_1 = \mathcal{C}$$
$$\mathcal{V}_i = \mathcal{C} \cap A^{-1}(\mathcal{V}_{i-1} + \mathcal{B}) \qquad i = 2, 3, \ldots$$

$\mathcal{V}^*_{(\mathcal{B},\mathcal{C})} = \mathcal{C} \cap A^{-1}(\mathcal{V}^*_{(\mathcal{B},\mathcal{C})} + \mathcal{B})$ is obtained when the sequence stops.

The maximum controlled invariant contained in $\mathcal{C}$

Consider the system

$$\Sigma : \quad \begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \qquad x(0) = x_0 \\ y(t) = C\,x(t) \end{cases}$$

Given a initial condition x_0 belonging to $\mathcal{V} \in \mathcal{V}(A, \operatorname{im} B, \ker C)$, we can find a control that maintains

- the state trajectory on $\mathcal{V}$ because $\mathcal{V}$ is controlled invariant
- the output at zero, because $\mathcal{V}$ is contained in ker C

For this reason, the subspaces in $\mathcal{V}(A, \operatorname{im} B, \ker C)$ are also called output-nulling for Σ.

$\mathcal{V}^*_{(\mathcal{B},\mathcal{C})}$ is the maximum locus of trajectories contained in $\mathcal{C}$: this means that a suitable control action can mantain the trajectory on $\mathcal{C}$ if and only if $x_0 \in \mathcal{V}^*_{(\mathcal{B},\mathcal{C})}$.

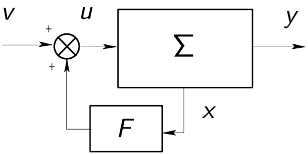
We will use the symbol $\mathcal{V}^*$ for $\max \mathcal{V}(A, \operatorname{im} B, \ker C)$: it is the maximal locus of controlled state trajectories such that the output is identically zero.

Reachable subspace on a controlled invariant

Given the triple (A, B, C), the $(A, \mathrm{im}\, B)$-controlled invariant $\mathcal{V}$ and the input function $u(t) = F\,x(t) + v(t)$:

$$\dot{x}(t) = (A + B\,F)\,x(t) + B\,v(t)\,, \quad x_0 \in \mathcal{V}$$
$$y(t) = C\,x(t)$$

the state trajectories belong to $\mathcal{V}$ if and only if $(A + B\,F)\,\mathcal{V} \subseteq \mathcal{V}$ and $v(t) \in B^{-1}\,\mathcal{V}\ \forall t$:



The reachable subspace from the origin by state trajectories constrained to belong to $\mathcal{V}$:

$$\mathcal{R}_{\mathcal{V}} = \min \mathcal{J}\,(A + B\,F,\ \mathcal{V} \cap \mathrm{im}\, B)$$

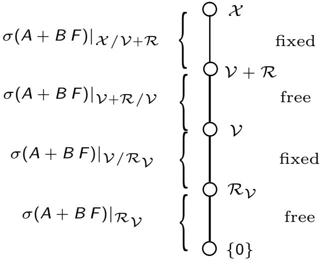
Being $(A + B\,F)$-invariant, $R_{\mathcal{V}}$ is an $(A,\ \mathrm{im}\, B)$ controlled invariant itself.

Outline

- Controlled invariance
- **Internal and external eigenvalues**
- Invariant zeros
- Disturbance decoupling problem

Internal and External Eigenvalues

Let $\mathcal{V}$ be an $(A, \mathcal{B})$-controlled invariant and F such that $(A + B\,F)\,\mathcal{V} \subseteq \mathcal{V}$. Being $\mathcal{R} = \min \mathcal{J}(A,\, \mathcal{B})$, we can partition the spectrum of $(A + B\,F)$ as in figure.



$\mathcal{V}$ is said to be

- internally stablizable if $\forall x_0 \in \mathcal{V}$, the trajectory can be mantained on $\mathcal{V}$ converging to the origin by a suitable control action; this happens if and only if $(A + B\,F)|_{\mathcal{V}}$ is stable, i.e. if and only if $(A + B\,F)|_{\mathcal{V}/R_\mathcal{V}}$ is stable
- externally stabilizable if $\forall x_0 \notin \mathcal{V}$, the trajectory converge to $\mathcal{V}$ by a suitable control action; this happens if and only if $(A + B\,F)|_{\mathcal{X}/\mathcal{V}}$ is stable, i.e. if and only if $(A + B\,F)|_{\mathcal{X}/\mathcal{V}+\mathcal{R}}$ is stable

Computation of feedback matrix F

Let V be a basis matrix of the $(A, \mathrm{im}\, B)$-controlled invariant $\mathcal{V}$. From equation $A\,V = V\,X + B\,U$ it is possible to derive a matrix F such that $(A + B\,F)\,\mathcal{V} \subseteq \mathcal{V}$:

$$\begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} V & B \end{bmatrix}^{+} A\,V + H\,K$$

where H is a basis matrix of $\ker \begin{bmatrix} V & B \end{bmatrix}$ and K is an arbitrary matrix of proper dimensions. Then compute

$$F = -U\,(V^T\,V)^{-1}\,V^T$$

The degree of freedom K allows to assign the internal assignable eigenstructure of $\mathcal{V}$.

Algorithm

Computation of the internal unassignable eigenstructure of an $(A, \mathcal{B})$-controlled invariant (A'_{22}). A matrix P representing the map $(A + BF)|_{\mathcal{V}/\mathcal{R}_{\mathcal{V}}}$ up to an isomorphism, is derived as follows. Consider $T = [T_1\ \ T_2\ \ T_3]$, with $\mathrm{im}\, T_1 = \mathcal{R}_{\mathcal{V}}$, $\mathrm{im}[T_1\ \ T_2] = \mathcal{V}$ and T_3 such that T is nonsingular. In the new basis matrix $A + BF$ is expressed by

$$(A + BF)' = T^{-1}(A + BF)\,T = \begin{bmatrix} A'_{11} & A'_{12} & A'_{13} \\ O & A'_{22} & A'_{23} \\ O & O & A'_{33} \end{bmatrix}$$

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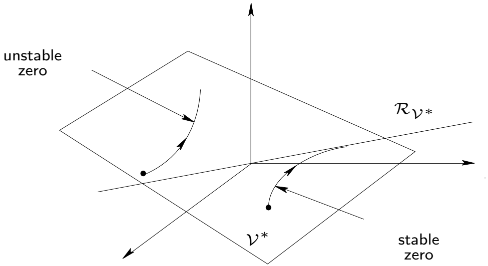
Invariant zeros of (A, B, C)

Roughly speaking, an invariant zero is a mode that, if suitably injected at the input of a system, can be nulled at the output by a suitable choice of the initial state.

Definition

The invariant zeros of (A, B, C) are the internal unassignable eigenvalues of $\mathcal{V}^*$. The invariant zero structure of (A, B, C) is the internal unassignable eigenstructure of $\mathcal{V}^*$.

Recall that $\mathcal{R}_{\mathcal{V}^*} = \min \mathcal{J}(A + BF, \mathcal{V}^* \cap \mathcal{B})$. The invariant zeros are the eigenvalues of the map $(A + BF)|_{\mathcal{V}^*/\mathcal{R}_{\mathcal{V}^*}}$, where F denotes any matrix such that $(A + BF)\mathcal{V}^* \subseteq \mathcal{V}^*$.



Invariant zeros of (A, B, C)

Consider the change of basis defined by transformation $T = [T_1, T_2, T_3]$ with $\mathrm{im}\, T_1 = \mathcal{R}_{\mathcal{V}^*}$, $\mathrm{im}[T_1\ T_2] = \mathcal{V}^*$, then

$$T^{-1}(A + BF)T = \begin{bmatrix} A'_{11} & A'_{12} & A'_{13} \\ 0 & A'_{22} & A'_{23} \\ 0 & 0 & A'_{33} \end{bmatrix}.$$

The invariant zeros of (A, B, C) are the eigenvalues of matrix A'_{22}.

`>> z = gazero(A,B,C,[D]);` Computation of invariant zeros (A, B, C) or (A, B, C, D)

The routines can be downloaded from the web page:
`http://www3.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm`

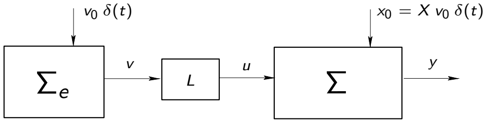
Invariant zeros of (A, B, C)

Let W be a real $m \times m$ matrix having the invariant zero structure of (A, B, C) as eigenstructure. A real $p \times m$ matrix L and a real $n \times m$ matrix X exist, with (W, X) observable, such that by applying to (A, B, C) the input function

$$u(t) = L\, e^{Wt}\, v_0 \tag{1}$$

where $v_0 \in \mathbb{R}^m$ denotes an arbitrary column vector, and starting from the initial state $x_0 = X\, v_0$, the output $y(\cdot)$ is identically zero, while the state evolution (on $\ker C$) is described by

$$x(t) = X\, e^{Wt}\, v_0 \tag{2}$$



Remark. *In the discrete-time case equations (1) and (2) are replaced by* $u(k) = LW^k\, v_0$ *and* $x(k) = XW^k\, v_0$*, respectively.*

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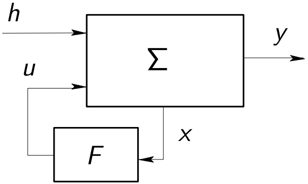
Disturbance Decoupling Problem

The *disturbance decoupling problem* is one of the earliest (1969) applications of the geometric approach.

Let us consider the system

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) + H\,h(t) \\ y(t) &= C\,x(t) \end{aligned}$$



where u denotes the manipulable input, h the disturbance input. Let $\mathcal{B} := \mathrm{im} B$, $\mathcal{H} := \mathrm{im} H$, $\mathcal{C} := \ker C$.

The *disturbance decoupling problem* is: "determine, if possible, a state feedback matrix F such that disturbance h has no influence on output y."
The system with state feedback is described by

$$\begin{aligned} \dot{x}(t) &= (A + B\,F)\,x(t) + H\,h(t) \\ y(t) &= C\,x(t) \end{aligned} \qquad (3)$$

It behaves as requested if and only if its reachable set by h, i.e., the minimum $(A + BF)$-invariant containing $\mathcal{H}$, is contained in $\mathcal{C}$.

Disturbance Decoupling Problem

Let $\mathcal{V}^*_{(\mathcal{B},\mathcal{C})} := \max \mathcal{V}(A,\mathcal{B},\mathcal{C})$. Since any $(A+BF)$-invariant is an $(A,\mathcal{B})$-controlled invariant, the inaccessible disturbance decoupling problem has a solution if and only if

$$\mathcal{H} \subseteq \mathcal{V}^*_{(\mathcal{B},\mathcal{C})} \tag{4}$$

Equation (4) is a *structural condition* and does not ensure internal stability. If stability is requested, we have the *disturbance decoupling problem with stability*. Stability is easily handled by using self-bounded controlled invariants.

Summary

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