Conditioned invariance and state observation

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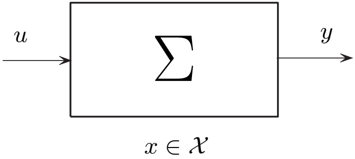
Block 1: Foundations [10:30 - 12.30]:

- Talk 1: *Motivation and historical perspective*, **G. Marro** [10:30 - 11:00]
- Talk 2: *Invariant subspaces*, **L. Ntogramatzidis** [11:00 - 11:30]
- Talk 3: *Controlled invariance and invariant zeros*, **D. Prattichizzo** [11:30 - 12:00]
- Talk 4: *Conditioned invariance and state observation*, **F. Morbidi** [12:00 - 12:30]

Block 2: Problems and applications [15:30 - 17.30]:

- Talk 5: *Stabilization and self-bounded subspaces*, **L. Ntogramatzidis** [15:30 - 16:00]
- Talk 6: *Disturbance decoupling problems*, **L. Ntogramatzidis** [16:00 - 16:30]
- Talk 7: *LQR and H_2 control problems*, **D. Prattichizzo** [16:30 - 17:00]
- Talk 8: *Spectral factorization and H_2-model following*, **F. Morbidi** [17:00 - 17:30]

Preliminaries



Continuous-time:

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) \end{aligned}$$

Discrete-time:

$$\begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) \end{aligned}$$

$x \in \mathbb{R}^n$ is the **state**, $u \in \mathbb{R}^p$ the **input** and $y \in \mathbb{R}^q$ the **output** of the system Σ.

Conditioned invariants

Definition

Given a linear map $A : \mathcal{X} \rightarrow \mathcal{X}$ and a subspace $\mathcal{C} \subseteq \mathcal{X}$, a subspace $\mathcal{S} \subseteq \mathcal{X}$ is an $(A, \mathcal{C})$-*conditioned invariant* if the following inclusion holds:

$$\boxed{A\,(\mathcal{S} \cap \mathcal{C}) \subseteq \mathcal{S}}$$

The conditioned invariants are **dual** to the controlled invariants.

Duality

Definition

Given a system $\Sigma : (A, B, C, D)$, we define its **dual system** as $\Sigma^T : (A^T, C^T, B^T, D^T)$ and we refer to it with the subscript "d".

- **Controllability** and **observability**:

$$\mathcal{R} = \mathcal{U}_d^{\perp}, \quad \mathcal{U} = \mathcal{R}_d^{\perp}.$$

  Therefore,

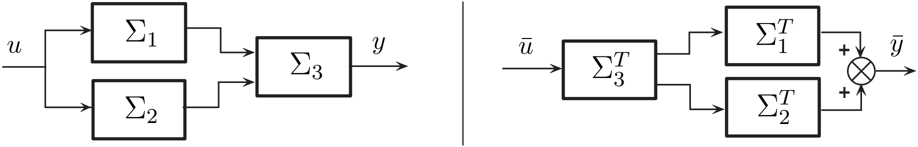
$$\Sigma \text{ completely controllable } \iff \Sigma^T \text{ completely observable}$$

  and vice versa.

- The orthogonal complement of an $(A, \mathcal{L})$-controlled (conditioned) invariant is an $(A^T, \mathcal{L}^{\perp})$-conditioned (controlled) invariant, i.e.,

$$\mathcal{V}^* = (\mathcal{S}_d^*)^{\perp}, \quad \mathcal{S}^* = (\mathcal{V}_d^*)^{\perp}.$$

Duality



Consider the connection of systems $\Sigma_i : (A_i, B_i, C_i)$, $i = 1, 2, 3$ in the figure. Denote the overall system with $\Sigma : (A, B, C)$, where:

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ B_{3,1}C_1 & B_{3,2}C_2 & A_3 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & C_3 \end{bmatrix}.$$

The **dual system** $\Sigma^T : (A^T, C^T, B^T)$ is obtained by reversing the order of serially connected systems and interchanging **branching points** with **summing junctions** and vice versa.

Properties of conditioned invariants

1. The **intersection** of any two $(A, \mathcal{C})$-conditioned invariants is an $(A, \mathcal{C})$-conditioned invariant, while their **sum** is not.

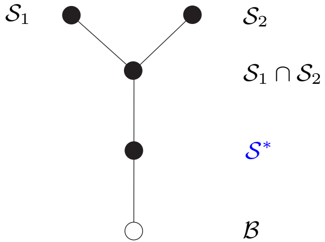
2. Let $\mathcal{B} \triangleq \text{im} B$, $\mathcal{C} \triangleq \ker C$. The state trajectories of the discrete-time system Σ that originate at $x(0) = 0$ can be made **invisible** at the output for a certain number of steps ρ while belonging to a given subspace $\mathcal{S}$ if and only if $\mathcal{S}$ is a $(A, \mathcal{C})$-conditioned invariant containing $\mathcal{B}$.

3. For any $(A, \mathcal{C})$-conditioned invariant $\mathcal{S}$ there is at least one matrix G, called a **friend** of $\mathcal{S}$, such that $(A + GC)\, \mathcal{S} \subseteq \mathcal{S}$.

   If a matrix G exists such that $\mathcal{S}$ is an externally stable and/or an internally stable $(A + GC)$-invariant, $\mathcal{S}$ is said to be **externally stabilizable** and/or **internally stabilizable**, respectively.

Comments on Property 1

The set of all the conditioned invariants containing $\mathcal{B} \subseteq \mathcal{X}$ (a *semilattice* with respect to $\subseteq$, $\cap$), admits an **infimum**, that coincides with their intersection.



It is computed through the sequence:

$$\begin{aligned} \mathcal{S}_1 &= \mathcal{B} \\ \mathcal{S}_i &= \mathcal{B} + A\,(\mathcal{S}_{i-1} \cap \mathcal{C}), \quad i = 2, 3, \ldots \end{aligned}$$

that converges to it in a *finite number* of steps.

Comments on Property 1

The previous sequence with $\mathcal{B} \triangleq \mathrm{im} B$, $\mathcal{C} \triangleq \ker C$, yields:

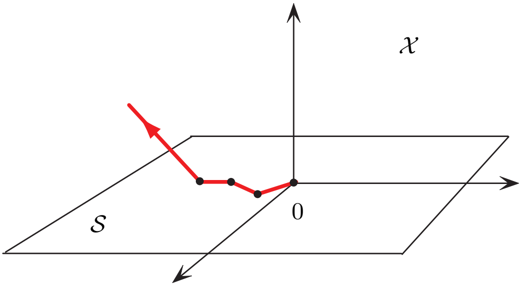
$\mathcal{S}^* =$ the minimal $(A, \mathcal{C})$ conditioned invariant containing $\mathcal{B}$
or **minimal input-containing conditioned invariant**.

For discrete-time systems, $\mathcal{S}^*$ is the *maximal subspace* of the state space reachable from the origin in at most ρ steps with trajectories that have all the states except the last one *invisible* at the output.

The value of ρ is the **number of steps** required for convergence of the sequence.

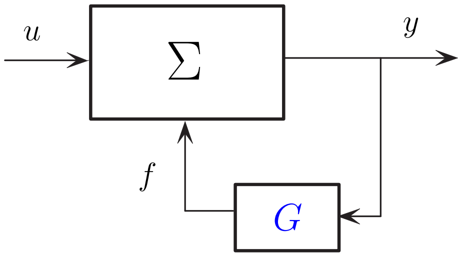
Comments on Property 2

Property 2 has no counterpart in *continuous time* unless control functions (typically, piecewise continuous), are extended to include **distributions**.



A state trajectory of a discrete-time system starting from the origin can be maintained on a subspace $\mathcal{S}$ for a certain number of steps with a suitable **control action** if and only if $\mathcal{S}$ is a conditioned invariant containing $\mathcal{B}$.

Comments on Property 3



This control action can be obtained by means of *output-to-state algebraic feedback* (**output injection**) where f denotes an input directly acting on the state.

A new insight into $\mathcal{R}^*$

$\mathcal{R}^* =$ *maximum subspace reachable from the origin with state trajectories in* $\mathcal{V}^*$

It has been shown by S. Morse that,

$$\boxed{\mathcal{R}^* = \mathcal{V}^* \cap \mathcal{S}^*}$$

- $\mathcal{R}^*$ can be computed by using the sequences for $\mathcal{V}^*$ and $\mathcal{S}^*$.
- $\mathcal{R}^*$ is an $(A, \mathcal{B})$-controlled invariant. In fact, the intersection of an $(A, \mathcal{B})$-controlled invariant contained in $\mathcal{C}$ and an $(A, \mathcal{C})$-conditioned invariant containing $\mathcal{B}$ is an $(A, \mathcal{B})$-controlled invariant.

A. S. Morse, "Structural invariants of linear multivariable systems", *SIAM J. Control*, vol. 11, no. 3, pp. 446–465, 1973.

Extension to quadruples

Consider the continuous and discrete-time systems with **feedthrough matrix** D:

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned} \qquad\qquad \begin{aligned} x(k+1) &= A\,x(k) + B\,u(k) \\ y(k) &= C\,x(k) + D\,u(k) \end{aligned}$$

- An **output-nulling subspace** and **its friend** are defined as a pair $(\mathcal{V}, F)$ such that

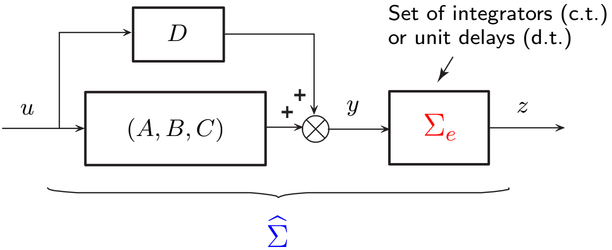
$$(A+BF)\mathcal{V} \subseteq \mathcal{V} \quad \text{and} \quad \mathcal{V} \subseteq \ker(C+DF).$$

- An **input-containing subspace** and **its friend** are defined as a pair $(\mathcal{S}, G)$ such that

$$(A+GC)\mathcal{S} \subseteq \mathcal{S} \quad \text{and} \quad \mathcal{S} \supseteq \operatorname{im}(B+GD).$$

Extension to quadruples

The computation of $\mathcal{V}^*$ and $\mathcal{S}^*$ is still possible with the previous sequences applied to the extended system $\widehat{\Sigma}$.



System $\widehat{\Sigma}$ can be described by the extended state $\hat{x}$ and extended triple $(\widehat{A}, \widehat{B}, \widehat{C})$:

$$\hat{x} = \begin{bmatrix} x \\ u \end{bmatrix}, \quad \widehat{A} = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} B \\ D \end{bmatrix}, \quad \widehat{C} = \begin{bmatrix} 0 & I_q \end{bmatrix}.$$

Extension to quadruples

Compute the output nulling $\widehat{\mathcal{V}}^*$ with basis matrix $\widehat{V}$ for system $\widehat{\Sigma}$ and a corresponding state feedback matrix $\widehat{F}$ and denote with

$$\widehat{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \quad \widehat{F} = \begin{bmatrix} F_1 & F_2 \end{bmatrix},$$

their partitions according to the triple $(\widehat{A}, \widehat{B}, \widehat{C})$.
Owing to the structure of $\widehat{C}$, it turns out that

$$V_2 = 0, \quad F_2 = 0,$$

and the maximum output nulling controlled invariant of the quadruple (A, B, C, D) is

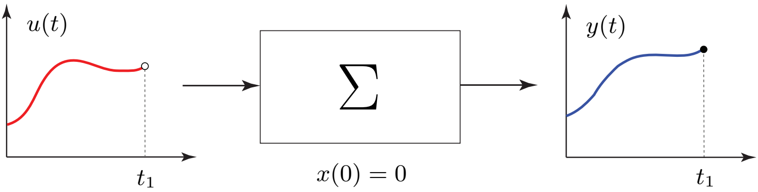
$$\mathcal{V}^* = \mathrm{im} V_1.$$

F_1 is a corresponding state feedback matrix adapted to its basis V_1.

A similar procedure applies to $\mathcal{S}^*$.

Left invertibility

Consider the **continuous-time** system $\Sigma : (A, B, C)$ or $\Sigma : (A, B, C, D)$ with $x(0) = 0$ and assume that $u(t)$ is bounded and piecewise continuous.



Definition

System Σ is said to be **left invertible** if, given any admissible output function $y(t)$, $t \in [0,\ t_1]$, $t_1 > 0$, there is a unique corresponding input function $u(t)$, $t \in [0,\ t_1)$ producing that output function.

Right invertibility and relative degree

Definition

System Σ is said to be **right invertible** if there is an integer $\rho \geq 1$ such that, given any output function $y(t)$, $t \in [0,\, t_1]$, $t_1 > 0$ with ρ-th derivative piecewise continuous and such that $y(0) = 0,\, \ldots,\, y^{(\rho-1)}(0) = 0$, there is at least one input function $u(t)$, $t \in [0,\, t_1)$ producing that output function.

Definition

The minimum value of ρ is called the (**global**) **relative degree** of Σ.

Remark

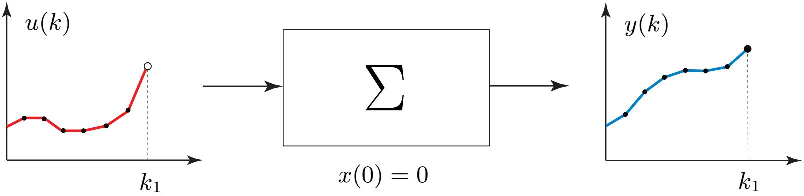
By duality,

$$\Sigma \text{ is left invertible } \iff \Sigma^T \text{ is right invertible}$$

and vice versa.

Left invertibility

Consider the **discrete-time** system $\Sigma : (A, B, C)$ or $\Sigma : (A, B, C, D)$ with $x(0) = 0$ and assume that its input function $u(k)$ is bounded.



Definition

The system is said to be **left invertible** if, given any admissible output function $y(k)$, $k \in [0,\ k_1]$, $k_1 \geq n$, there is a unique corresponding input function $u(k)$, $k \in [0,\ k_1)$ producing that output function.

Right invertibility and relative degree

Definition

The system is said to be **right invertible** if there is an integer $\rho \geq 1$ such that, given an output function $y(k)$, $k \in [0, k_1]$, $k_1 \geq \rho$ such that $y(k) = 0$, $k \in [0, \rho - 1]$, there is at least one input function $u(k)$, $k \in [0, k_1 - 1]$ producing that output function.

Definition

The minimum value of ρ is called the (**global**) **relative degree** of the system.

Properties expressed in geometric terms

Geometric necessary and sufficient conditions for left and right invertibility:

- **Left invertibility:**

$$\boxed{\mathcal{V}^* \cap \mathcal{S}^* = \{0\}}$$

- **Right invertibility:**

$$\boxed{\mathcal{V}^* + \mathcal{S}^* = \mathcal{X}}$$

- **Relative degree:**

  For a right invertible system with $D = 0$, the relative degree is the minimal value of ρ such that,

$$\mathcal{V}^* + \mathcal{S}_\rho = \mathcal{X}$$

  where $\mathcal{S}_i$, $i = 1, 2, \ldots$ are provided by the previously defined sequence.

Computational tools

Computational routines of the **Geometric Approach (GA) toolbox** for MATLAB:

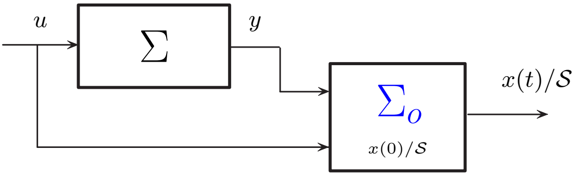
```
>> [S,G] = sstar(A,B,C,[D]);
```
Computation of $\mathcal{S}^*$ and a friend G

```
>> rho = reldeg(A,B,C,[D]);
```
Computation of the relative degree ρ

The routines can be downloaded from the web page:

http://www3.deis.unibo.it/Staff/FullProf/GiovanniMarro/geometric.htm

Conditioned invariants and state observation



The notion of conditioned invariance is closely connected with "*maintaining information*" on the state vector of an observed linear system $\Sigma : (A, B, C)$.

Problem

Assume that $\mathcal{S} \subseteq \mathcal{X}$ and that we know $x(0)/\mathcal{S}$. Does there exist a "mechanism" that, using exact knowledge of $x(0)/\mathcal{S}$ together with the observations $\{u(\tau), y(\tau)\}$, $0 \leq \tau < t$, provides $x(t)/\mathcal{S}$?

Such "mechanism" is called an **observer** Σ_o for $x/\mathcal{S}$.

Conditioned invariants and state observation

Definition

An observer Σ_o for $x/\mathcal{S}$ is a system,

$$\dot{w}(t) = P\,w(t) + Q\,u(t) + R\,y(t)$$
$$\zeta(t) = S\,w(t)$$

with output space $\mathcal{X}/\mathcal{S}$, such that for each pair of initial conditions $(x(0),\, w(0))$ and any input function u we have:

$$\zeta(0) = x(0)/\mathcal{S} \quad \Rightarrow \quad \zeta(t) = x(t)/\mathcal{S}, \;\; \forall\, t \geq 0.$$

Problem: If $\mathcal{S} \subseteq \mathcal{X}$, does there *always* exist an observer for $x/\mathcal{S}$? **No !!**

Definition

$\mathcal{S} \subseteq \mathcal{X}$ is a **conditioned invariant** if there exists an observer for $x/\mathcal{S}$.

Conditioned invariants and state observation

- Assume that $x(0)/\mathcal{S}$ is **not known**. Apart from the *information-maintaining property*, a desired property of an observer would be that for any input u and for any pair $(x(0), w(0))$:

$$\lim_{t\to\infty} \zeta(t) - x(t)/\mathcal{S} = 0.$$

- An observer for $x/\mathcal{S}$ that has this property is called **asymptotic**.

Definition

$\mathcal{S} \subseteq \mathcal{X}$ is an **externally stabilizable conditioned invariant** if there exists an asymptotic observer for $x/\mathcal{S}$.

Summary

- Conditioned invariant subspaces $\mathcal{S}$
- Duality
- Properties of conditioned invariants
- The minimum input-containing conditioned invariant $\mathcal{S}^*$
- $\mathcal{R}^*$ and its relation to $\mathcal{V}^*$ and $\mathcal{S}^*$
- From triples (A, B, C) to quadruples (A, B, C, D)
- Left, right invertibility and relative degree
- Conditioned invariants and state observation