Curtin
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Stabilization and Self bounded Subspaces

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Block 1: Foundations [10:30 - 12.30]:

- Talk 1: *Motivation and historical perspective*, **G. Marro** [10:30 - 11:00]
- Talk 2: *Invariant subspaces*, **L. Ntogramatzidis** [11:00 - 11:30]
- Talk 3: *Controlled invariance and invariant zeros*, **D. Prattichizzo** [11:30 - 12:00]
- Talk 4: *Conditioned invariance and state observation*, **F. Morbidi** [12:00 - 12:30]

Block 2: Problems and applications [15:30 - 17.30]:

- Talk 5: *Stabilization and self-bounded subspaces*, **L. Ntogramatzidis** [15:30 - 16:00]
- Talk 6: *Disturbance decoupling problems*, **L. Ntogramatzidis** [16:00 - 16:30]
- Talk 7: *LQR and H_2 control problems*, **D. Prattichizzo** [16:30 - 17:00]
- Talk 8: *Spectral factorization and H_2-model following*, **F. Morbidi** [17:00 - 17:30]

Outline

- **Stabilisation of Controlled Invariant Subspaces**
- Self bounded subspaces
- Disturbance decoupling problem (DDP)

Controlled invariant and output nulling subspaces

$$\Sigma: \quad \begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) & x(0) = x_0 \\ y(t) = C\,x(t) \end{cases}$$

$$\boxed{\mathcal{B} \stackrel{\text{def}}{=} \operatorname{im} B \quad \text{and} \quad \mathcal{C} \stackrel{\text{def}}{=} \ker C}$$

Controlled invariant subspaces are loci of trajectories for Σ:

- if $x_0 \in \mathcal{V}$, we can find $u(\cdot)$ such that $x(t) \in \mathcal{V}$ for all $t \geq 0$;
- the subspace of minimal dimension containing a trajectory $x(\cdot)$ is controlled invariant.

Output nulling subspaces are controlled invariants contained in $\ker C$.

- $\mathcal{V}^\star_{(\mathcal{B},\mathcal{C})}$ is the largest output-nulling subspace: if we want $x(\cdot)$ to yield $y = 0$, we need $x(t) \in \mathcal{V}^\star_{(\mathcal{B},\mathcal{C})}$ $\forall\, t$.

Friends

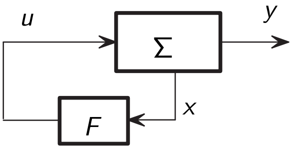
Given a controlled invariant $\mathcal{V}$ and $x_0 \in \mathcal{V}$,

- a control u exists that maintains the state trajectory on $\mathcal{V}$
- such control can always be expressed as a static feedback

$$u(t) = F\,x(t)$$

where F is a *friend* of $\mathcal{V}$, i.e., $(A + B\,F)\mathcal{V} \subseteq \mathcal{V}$:

$$\dot{x}(t) = (A + B\,F)\,x(t) \qquad x(0) \in \mathcal{V} \qquad \Longrightarrow \qquad x(t) \in \mathcal{V} \;\; \forall\, t \geq 0$$



A Trivial Friend

- $A\,\mathcal{V} \subseteq \mathcal{V} + \mathcal{B}$ means

$$\exists X,\, U: \qquad A\,V = V\,X + B\,U$$

  where V is a basis of $\mathcal{V}$;
- let F be such that $U = -F\,V$;
- then

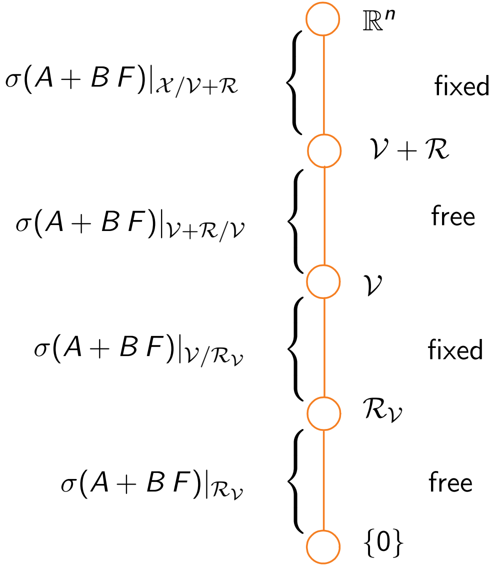
$$(A + B\,F)\,V = V\,X$$

  which means $(A + B\,F)\,\mathcal{V} \subseteq \mathcal{V}$.

We are not exploiting 2 degrees of freedom:

- in the solution of $A\,V = V\,X + B\,U$
- in the solution $U = -F\,V$

Friends


$\mathbb{R}^n$
$\sigma(A + B\,F)|_{\mathcal{X}/\mathcal{V}+\mathcal{R}}$
fixed
$\mathcal{V} + \mathcal{R}$
$\sigma(A + B\,F)|_{\mathcal{V}+\mathcal{R}/\mathcal{V}}$
free
$\mathcal{V}$
$\sigma(A + B\,F)|_{\mathcal{V}/\mathcal{R}_{\mathcal{V}}}$
fixed
$\mathcal{R}_{\mathcal{V}}$
$\sigma(A + B\,F)|_{\mathcal{R}_{\mathcal{V}}}$
free
$\{0\}$

Friends - Linear equations

- Equation

$$\boxed{M\,X = N}$$

admits solutions if and only if $\mathrm{im}\, N \subseteq \mathrm{im}\, M$. The set of solutions is

$$\{X = M^{+}\,N + H\,K \mid \mathrm{im}\, H = \ker\, M \text{ and } K \text{ is arbitrary}\}$$

- Equation

$$\boxed{X\,M = N}$$

admits solutions if and only if $\ker N \supseteq \ker M$. The set of solutions is

$$\{X = N\,M^{+} + K\,H \mid \ker H = \mathrm{im}\, M \text{ and } K \text{ is arbitrary}\}$$

Friends

- All solutions of $A\,V = V\,X + B\,U$ (or $A\,V = [\,V \quad B\,]\begin{bmatrix} X \\ U \end{bmatrix}$) are

$$\begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} V & B \end{bmatrix}^{+} A\,V + H_1\,K_1$$

  where $\mathrm{im}\,H_1 = \ker \begin{bmatrix} V & B \end{bmatrix}$ and K_1 is arbitrary;
- The set of solutions of $U = -F\,V$ is given by

$$F = -U\,(V^\top V)^{-1}\,V^\top + K_2\,H_2$$

  where $\ker H_2 = \mathcal{V}$ and K_2 is arbitrary.

It is easy to show that

- K_1 only affects the internal eigenvalues of $\mathcal{V}$
- K_2 only affects the external eigenvalues of $\mathcal{V}$

Friends - Internal Stabilization

$\mathbb{R}^n$

$\sigma(A + B\,F)|_{\mathcal{X}/\mathcal{V}+\mathcal{R}}$ — fixed

$\mathcal{V} + \mathcal{R}$

$\sigma(A + B\,F)|_{\mathcal{V}+\mathcal{R}/\mathcal{V}}$ — free

$\mathcal{V}$

$\sigma(A + B\,F)|_{\mathcal{V}/\mathcal{R}_\mathcal{V}}$ — fixed

$\mathcal{R}_\mathcal{V}$

$\sigma(A + B\,F)|_{\mathcal{R}_\mathcal{V}}$ — free

$\{0\}$

Friends - Internal stabilization

Given $\mathcal{V}$ and a basis V, the reachable subspace on $\mathcal{V}$ is given by

$$\mathcal{R}_{\mathcal{V}} = \mathcal{V} \cap \mathcal{S}_{\mathcal{V}}^{\star}$$

where $\mathcal{S}_{\mathcal{V}}^{\star}$ is given by

$$\begin{cases} \mathcal{S}_1 = \operatorname{im} B \\ \mathcal{S}_i = \operatorname{im} B + A\,(\mathcal{S}_{i-1} \cap \mathcal{V}) \qquad i = 2, 3, \ldots \end{cases}$$

Now, we use a basis $V = [\,R_V \;\; V_c\,]$ for $\mathcal{V}$ such that $\operatorname{im} R_V = \mathcal{R}_{\mathcal{V}}$.
We can write $\begin{bmatrix} X \\ U \end{bmatrix} = \begin{bmatrix} V & B \end{bmatrix}^{+} A\,V + H_1\,K_1$ as

$$\left[\begin{array}{cc} X_{11} & X_{12} \\ O & X_{22} \\ \hline U_1 & U_2 \end{array}\right] = \begin{bmatrix} R_V & V_c & B \end{bmatrix}^{+} A \begin{bmatrix} R_V & V_c \end{bmatrix} + \left[\begin{array}{c} H_1 \\ O \\ \hline H_3 \end{array}\right] \begin{bmatrix} K_1' & K_1'' \end{bmatrix}$$

Friends - Internal stabilization

Hence

$$\left[\begin{array}{cc} X_{11} & X_{12} \\ O & X_{22} \\ \hline U_1 & U_2 \end{array}\right] = \left[\begin{array}{cc} \Xi_{11} & \Xi_{12} \\ O & \Xi_{22} \\ \Omega_1 & \Omega_2 \end{array}\right] + \left[\begin{array}{c} H_1 \\ O \\ \hline H_2 \end{array}\right] \left[\begin{array}{cc} K_1' & K_1'' \end{array}\right]$$

which means

$$\left[\begin{array}{cc} X_{11} & X_{12} \\ O & X_{22} \\ \hline U_1 & U_2 \end{array}\right] = \left[\begin{array}{cc} \Xi_{11} + H_1\, K_1' & \Xi_{12} + H_1\, K_1'' \\ O & \Xi_{22} \\ \hline \Omega_1 + H_3\, K_1' & \Omega_2 + H_3\, K_1'' \end{array}\right]$$

- (Ξ_{11}, H_1) is controllable $\Longrightarrow$ K_1' can place all the spectrum of $\Xi_{11} + H_1\, K_1'$.

Friends - Internal stabilization

Assignment of internal dynamics using GA for MATLAB®:

>> Rv=ints(V,miinco(A,V,B));

>> r=size(V,2); q=size(Rv,2);

>> V=ima([Rv V],0);

>> XU=pinv([V B])*A*V;

>> H=ker([V B]);

>> Ξ_{11}=XU(1:q,1:q);

>> H_1=H(1:q,:);

>> p=[λ_1 λ_1 ...λ_q] (λ_i arbitrary)

>> K_1=-place(Ξ_{11},H_1,p);

>> XU(:,1:q)=XU(:,1:q)+H*K_1;

>> L=XU(r+1:r+m,:);

>> F_1=-L*pinv(V);

Friends - External Stabilization

$\mathbb{R}^n$

$\sigma(A+B\,F)|_{\mathcal{X}/\mathcal{V}+\mathcal{R}}$ — fixed

$\mathcal{V}+\mathcal{R}$

$\sigma(A+B\,F)|_{\mathcal{V}+\mathcal{R}/\mathcal{V}}$ — free

$\mathcal{V}$

$\sigma(A+B\,F)|_{\mathcal{V}/\mathcal{R}_\mathcal{V}}$ — fixed

$\mathcal{R}_\mathcal{V}$

$\sigma(A+B\,F)|_{\mathcal{R}_\mathcal{V}}$ — free

$\{0\}$

Friends - External stabilization

Changing coordinates of $(A+B\,F_1,B)$ with $T=\left[\begin{array}{ccc} T_1 & T_2 & T_3 \end{array}\right]$ such that

- $\mathrm{im}\, T_1 = \mathcal{V}$
- $\left[\begin{array}{cc} T_1 & T_2 \end{array}\right] = \mathcal{V}+\mathcal{R}$ where $\mathcal{R} = \min J(A, \mathrm{im}\, B)$

leads to

$$\bar{A} = T^{-1}(A+BF_1)\,T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ O & A_{22} & A_{23} \\ O & O & A_{33} \end{bmatrix}, \quad \bar{B} = T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \\ O \end{bmatrix}, \quad \bar{F} = F_1\,T$$

- O are due to $\mathcal{V}$ being $(A+B\,F_1)$-invariant;
- O are due to $\mathcal{V}+\mathcal{R}$ being A-invariant:

$$A\,\mathcal{V} \subseteq \mathcal{V}+\mathcal{B}, \quad A\,\mathcal{R} \subseteq \mathcal{R}, \quad \mathcal{R} \supseteq \mathcal{B} \implies A(\mathcal{V}+\mathcal{R}) \subseteq \mathcal{V}+\mathcal{B}+\mathcal{R} = \mathcal{V}+\mathcal{R}$$

- (A_{22}, B_2) is controllable.

Friends - External stabilization

Assignment of external dynamics with GA. Construction of T:

```
>> R=mininv(A,B);
>> T1=V; c1=size(T1,2);
>> c=size(ima([T1,R],0),2);
>> T1T2=ima([T1,R],0);
>> if c>=c1+1, T2=T1T2(:,c1+1:c); c2=size(T2,2);
>>   else T2=[]; c2=0;
>> end
>> if c<n, T3=ortco([T1 T2]);
>>   if  any(T3), T3=[]; end
>>   c3=size(T3,2);
>>   T=[T1 T2 T3];
>>   else c3=0; T=[T1 T2];
>> end
```

Friends - External stabilization

Assignment of external dynamics with GA. Construction of F:

```
>> if c2==0,
>>   F=F1;
>> else
>>   Ap=inv(T)*(A+B*F1)*T; Bp=inv(T)*B; Fp=F1*T;
>>   As=Ap(c1+1:c1+c2,c1+1:c1+c2);
>>   Bs=Bp(c1+1:c1+c2,:);
>>   p=[λ1  λ1  ...λc2]      (λi arbitrary)
>>   Fp(:,c1+1:c1+c2)=-place(As,Bs,p);
>>   F=Fp*inv(T);
>> end
```

Self-bounded subspaces

Self-bounded subspaces are particular output-nulling subspaces.

$$\Sigma : \quad \begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) & x(0) = x_0 \\ y(t) = C\,x(t) \end{cases}$$

Let $\mathcal{V}$ be output-nulling and a *friend* F s.t. $(A + B\,F)\,\mathcal{V} \subseteq \mathcal{V} \subseteq \ker C$.
Suppose we want:

- to "escape" $\mathcal{V}$
- to remain in $\ker C$, and therefore in $\mathcal{V}^\star_{(\mathcal{B},\mathcal{C})} \quad \Longrightarrow \quad y = 0.$

The set of state velocities that can keep us in $\ker C$ is

$$\mathcal{T}(x(t)) = (A + B\,F)\,x(t) + (\mathcal{V}^\star_{(\mathcal{B},\mathcal{C})} \cap \operatorname{im} B)$$

If $x_0 \in \mathcal{V}$ and $\mathcal{V} \supseteq \mathcal{V}^\star_{(\mathcal{B},\mathcal{C})} \cap \operatorname{im} B$, we cannot escape $\mathcal{V}$, unless we leave $\ker C$. In this case, $\mathcal{V}$ is called **self-bounded**.

Self-bounded subspaces: Properties

Definition

Let $\mathcal{V}$ be an output-nulling for (A, B, C) and let $\mathcal{V}^\star = \max \mathcal{V}(A, \mathcal{B}, \mathcal{C})$. Then, $\mathcal{V}$ is said to be self-bounded if

$$\boxed{\mathcal{V} \supseteq \mathcal{V}^\star_{(\mathcal{B},\mathcal{C})} \cap \mathrm{im}\, B}$$

We define

$$\Phi(\mathcal{B}, \mathcal{C}) = \{\mathcal{V} \in \mathcal{V}(A, \mathcal{B}, \mathcal{C}) \mid \mathcal{V} \supseteq \mathcal{V}^\star_{\mathcal{B},\mathcal{C}} \cap \mathrm{im}\, B\}$$

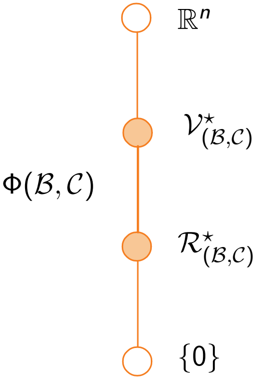
If $\mathcal{V} \in \Phi(\mathcal{B}, \mathcal{C})$, then $\mathcal{V}$ cannot be exited by means of any trajectory on $\mathcal{C}$. Trivially:

- $\mathcal{V}^\star_{(\mathcal{B},\mathcal{C})}$ is self-bounded
- $\mathcal{R}^\star_{(\mathcal{B},\mathcal{C})}$ is self-bounded

Self-bounded subspaces: Properties

Differently from $\mathcal{V}(A,\mathcal{B},\mathcal{C})$, the set $\Phi(\mathcal{B},\mathcal{C})$ is closed under intersection.

- Its maximum is $\mathcal{V}^\star_{(\mathcal{B},\mathcal{C})}$
- Its minimum is $\mathcal{R}^\star_{(\mathcal{B},\mathcal{C})} = \mathcal{V}^\star_{(\mathcal{B},\mathcal{C})} \cap \mathcal{S}^\star_{(\mathcal{C},\mathcal{B})}$.



Self-Hidden Subspaces

Using

$$
\begin{array}{rcl}
(\mathcal{X}+\mathcal{Y})^\perp & = & \mathcal{X}^\perp \cap \mathcal{Y}^\perp \\
(\mathcal{X}\cap\mathcal{Y})^\perp & = & \mathcal{X}^\perp + \mathcal{Y}^\perp \\
A\mathcal{X} \subseteq \mathcal{H} & \Leftrightarrow & A^\top \mathcal{H}^\perp \subseteq \mathcal{X}^\perp \\
(\mathrm{im}\, A)^\perp & = & \ker\, A^\top
\end{array}
$$

it is found that

- $\mathcal{V}$ is controlled invariant for (A,B,C) iff $\mathcal{V}^\perp$ is conditioned invariant for $(A^\top, C^\top, B^\top)$;
- $\Big(\max\, \mathcal{V}(A, \mathrm{im}\, B, \ker\, C)\Big)^\perp = \min \mathcal{S}(A^\top, \ker\, B^\top, \mathrm{im}\, C^\top)$

$$
\begin{aligned}
\mathcal{V} \supseteq \mathcal{V}^\star_{\mathcal{B},\mathcal{C}} \cap \mathrm{im}\, B \;&\Longrightarrow\; \mathcal{V}^\perp \subseteq (\mathcal{V}^\star_{\mathcal{B},\mathcal{C}} \cap \mathrm{im}\, B)^\perp \\
&\Longrightarrow\; \mathcal{V}^\perp \subseteq (\mathcal{V}^\star_{\mathcal{B},\mathcal{C}})^\perp + (\mathrm{im}\, B)^\perp \\
&\Longrightarrow\; \mathcal{V}^\perp \subseteq \min\, S(A^\top, \ker\, B^\top, \mathrm{im}\, C^\top) + \ker\, B^\top
\end{aligned}
$$

Self-Hidden Subspaces

Definition

Let $\mathcal{S}$ be an input-containing for (A, B, C) and let $\mathcal{S}^\star = \min \mathcal{S}\,(A, \mathcal{C}, \mathcal{B})$. Then, $\mathcal{S}$ is said to be self-hidden if

$$\boxed{\mathcal{S} \subseteq \mathcal{S}^\star_{(\mathcal{C},\mathcal{B})} + \ker C}$$

We define

$$\Psi(\mathcal{C}, \mathcal{B}) = \{\mathcal{S} \in \mathcal{S}\,(A, \mathcal{C}, \mathcal{B}) \mid \ \mathcal{S} \subseteq \mathcal{S}^\star_{\mathcal{C},\mathcal{B}} + \ker C\}$$

Exploiting duality:

- $\mathcal{S}^\star_{(\mathcal{C},\mathcal{B})}$ is self-hidden
- $\mathcal{V}^\star_{(\mathcal{B},\mathcal{C})} + \mathcal{S}^\star_{(\mathcal{C},\mathcal{B})}$ is self-hidden

Self-hidden subspaces: Properties

Differently from $\mathcal{S}\,(A, \mathcal{C}, \mathcal{B})$, the set $\Psi(\mathcal{B}, \mathcal{C})$ is closed under sum.

- Its maximum is $\mathcal{S}^{\star}_{(\mathcal{C},\mathcal{B})}$
- Its minimum is $\mathcal{S}^{\star}_{(\mathcal{C},\mathcal{B})} + \mathcal{V}^{\star}_{(\mathcal{B},\mathcal{C})}$.