LQR and H_2 control problems

Domenico Prattichizzo

DII, University of Siena, Italy

MTNS - July 5-9, 2010

Block 1: Foundations [10:30 - 12.30]:

- Talk 1: *Motivation and historical perspective*, **G. Marro** [10:30 - 11:00]
- Talk 2: *Invariant subspaces*, **L. Ntogramatzidis** [11:00 - 11:30]
- Talk 3: *Controlled invariance and invariant zeros*, **D. Prattichizzo** [11:30 - 12:00]
- Talk 4: *Conditioned invariance and state observation*, **F. Morbidi** [12:00 - 12:30]

Block 2: Problems and applications [15:30 - 17.30]:

- Talk 5: *Stabilization and self-bounded subspaces*, **L. Ntogramatzidis** [15:30 - 16:00]
- Talk 6: *Disturbance decoupling problems*, **L. Ntogramatzidis** [16:00 - 16:30]
- Talk 7: *LQR and H_2 control problems*, **D. Prattichizzo** [16:30 - 17:00]
- Talk 8: *Spectral factorization and H_2-model following*, **F. Morbidi** [17:00 - 17:30]

Outline

- Statement of the problem
- Hamiltonian formulation
- Geometric insight in the Hamiltonian system
- Solving the cheap LQ problem in a geometric setting

LQR problems

Consider

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) \qquad x(0) = x_0 \\ y(t) &= C\,x(t) + D\,u(t) \end{aligned}$$

and the index to be minimized

$$J(x,u) = \int_0^\infty y^\top(t)\,y(t)dt = \int_0^\infty \begin{bmatrix} x^\top(t) & u^\top(t) \end{bmatrix} \begin{bmatrix} C^\top C & C^\top D \\ D^\top C & D^\top D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

$$= \int_0^\infty \left(x^\top(t)\,Q\,x(t) + 2\,x^\top(t)\,S\,u(t) + u^\top(t)\,R\,u(t) \right)\,dt$$

with $Q = C^\top C$, $S = C^\top D$ and $R = D^\top D$. Usually $S = 0$.
The problem is

Regular when $R > 0 \leftrightarrow \ker D = \{0\}$

Singular when $R \geq 0 \leftrightarrow \ker D \supset \{0\}$

Cheap when $R = 0 \leftrightarrow \ker D = \mathbb{R}^m$

The main contributions

LQR in a geometric setting

- The LQR problem for singular cases was investigated by Hautus, Silvermann, Willems using *distributions*.
- Saberi and Sannuti (1987) investigated the cheap ($R = 0$) and singular ($R \geq 0$) problems with the *special coordinate basis*.
- Saberi, Sannuti and Stoorvogel (1992 and 1995) proposed a solution based on LMIs.

Contribution

The geometric techniques are applied directly to the Hamiltonian system.
The cheap LQ problem admits solution with an algebraic feedback only if the initial condition is contained in a special subspace.
This talk is based on Prattichizzo, Ntogramatzidis and G. Marro (Automatica, 2008)

Statement of the problem - Cheap LQ problem

Consider the LTI system Σ

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t)\,, \quad x(0) = x_0 \\ y(t) &= C\,x(t) \end{aligned}$$

with the assumptions:

(A1) (A, B) is stabilizable

(A2) Σ has no invariant zeros on $j\,\mathbb{R}$

Determine the set of initial conditions x_0 such that a static state feedback matrix K exists s.t.

- $A - B\,K$ is stable
- the corresponding state trajectory minimizes

$$J(x, u) = \frac{1}{2}\int_0^\infty y^\top(t)\,y(t)dt = \frac{1}{2}\int_0^\infty x^\top(t)C^\top C\,x(t)\,dt$$

Hamiltonian formulation

Consider the Hamiltonian function

$$H(t) := \frac{1}{2}x^{\mathsf{T}}(t)C^{\mathsf{T}}C\,x(t) + \lambda^{\mathsf{T}}(t)\,(A\,x(t) + B\,u(t))$$

and derive the state, costate equations and stationary condition as

$$\begin{aligned}
\dot{x}(t) &= \left(\frac{\partial H(t)}{\partial \lambda(t)}\right)^{\mathsf{T}} = A\,x(t) + B\,u(t) \\
\dot{\lambda}(t) &= \left(\frac{\partial H(t)}{\partial x(t)}\right)^{\mathsf{T}} = -C^{\mathsf{T}}C\,x(t) - A^{\mathsf{T}}\lambda(t) \\
0 &= \left(\frac{\partial H(t)}{\partial u(t)}\right)^{\mathsf{T}} = B^{\mathsf{T}}\lambda(t)
\end{aligned}$$

The Hamiltonian system

The optimal state trajectory and control law for our problem satisfy the *Hamiltonian system*:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C^\top C & -A^\top \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t)$$

$$\tilde{y}(t) = \begin{bmatrix} 0 & B^\top \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = 0$$

The matrices above are denoted by $\widehat{A}$, $\widehat{B}$ and $\widehat{C}$, while $(\widehat{A}, \widehat{B}, \widehat{C})$ is denoted by $\widehat{\Sigma}$.

Problem: find $u(t) = K\,x(t)$ such that for $x(0) = x_0$

- $\tilde{y}(t) = 0$ for all $t \geq 0$
- $x(t) \to 0$ for $t \to \infty$.

Geometric approach: notations and properties

The geometric approach setting will require the following notations

- $\mathcal{V}_{\Sigma}^{*}$: the maximum $(A, \mathrm{im} B)$-controlled invariant subspace of (A, B, C) contained in the null space of C.
- $\mathcal{S}_{\Sigma}^{*}$: the minimum $(A, \ker C)$-conditioned invariant subspace containing the image of B.

F is a *friend* of $\mathcal{V}_{\Sigma}^{*}$ if

$$(A + B\,F)\,\mathcal{V}_{\Sigma}^{*} \subseteq \mathcal{V}_{\Sigma}^{*}$$

At least one friend exists for every controlled invariant.

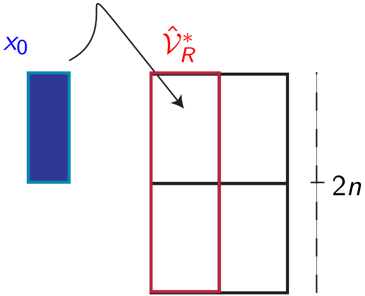
Geometric approach

The Hamiltonian system is

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{A}\,\hat{x}(t) + \hat{B}\,u(t) \\ 0 &= \hat{C}\,\hat{x}(t) \end{aligned} \qquad \hat{x} = \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

The problem is **equivalent** to a disturbance decoupling problem in a state-costate domain.

The cheap LQR has solutions iff an internally stabilizable output nulling subspace of $\widehat{\Sigma}$ exists whose projection on the state space of Σ contains x_0.



Classical DDP conditions

- $\hat{\mathcal{V}}_R^* \subseteq \ker \hat{C}$ and
- modes on $\hat{\mathcal{V}}_R^*$ must be stable.

Solving the cheap LQR problem

1. compute $\mathcal{V}^*_{\hat{\Sigma}}$;
2. compute a matrix $\hat{F}$ such that $(\hat{A}+\hat{B}\hat{F})\mathcal{V}^*_{\hat{\Sigma}} \subseteq \mathcal{V}^*_{\hat{\Sigma}}$
3. compute $\hat{\mathcal{V}}_R$, the maximum internally stable $(\hat{A}+\hat{B}\hat{F})$-invariant contained in $\mathcal{V}^*_{\hat{\Sigma}}$;
4. if $x_0 \in \mathcal{P}(\hat{\mathcal{V}}_R)$ the problem admits a solution F; if not, the problem has no solution.

Next

What about the geometric structure and dimension of $\hat{\mathcal{V}}_R$?

Further notation

Σ is left invertible iff $\mathcal{V}^*_\Sigma \cap \mathcal{S}^*_\Sigma = \{0\}$.

Σ is right invertible iff $\mathcal{V}^*_\Sigma + \mathcal{S}^*_\Sigma = \mathbb{R}^n$.

Adjoint system

Let $\Sigma^\top = (A^\top, C^\top, B^\top)$ be the *adjoint* of $\Sigma = (A, B, C)$.

- Σ is left invertible iff $\Sigma^\top$ is right invertible
- Σ is right invertible iff $\Sigma^\top$ is left invertible

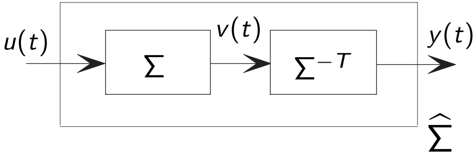
Time-reversed system

$\Sigma^{-1} = (-A, -B, C)$ is the *time-reversed system* associated with Σ.

- $\mathcal{V}^*_\Sigma = \mathcal{V}^*_{\Sigma^{-1}}$ and $\mathcal{S}^*_\Sigma = \mathcal{S}^*_{\Sigma^{-1}}$
- Σ is left invertible iff Σ^{-1} is left invertible
- Σ is right invertible iff Σ^{-1} is right invertible

A geometric insight in the Hamiltonian system

$\widehat{\Sigma}$ is the series connection of Σ and Σ^{-T}

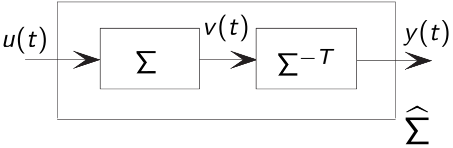


$\boxed{\Sigma \text{ left invertible}} \Longrightarrow \hat{\Sigma}$ is left and right invertible.

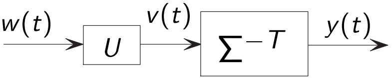
- Left invertibility of $\widehat{\Sigma}$ [IEEE TAC, 2002].
- Right invertibility of $\widehat{\Sigma}$ comes from equivalence of $\widehat{\Sigma}^{\mathsf{T}}$ and $\widehat{\Sigma}^{-1}$ through a coordinate transformation.

The set of **invariant zeros** of $\widehat{\Sigma}$ consists of all the invariant zeros of $\widehat{\Sigma}$ along with their opposite, hence they are all coupled by pairs $(z, -z)$.

How the second block is functionally driven

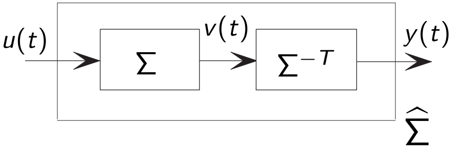


Let U be a basis matrix of the subspace of *functional controllability* $C\,\mathcal{S}^*_\Sigma$, and denote by the symbol $\bar{\Sigma}$ the new system $(A^T, C^T U, B^T)$.



System $\bar{\Sigma}$ is left and right invertible, and $\mathcal{V}^*_{\bar{\Sigma}} = \mathcal{V}^*_{\Sigma^{-T}} = \mathcal{V}^*_{\Sigma^T}$ and
$\dim \mathcal{S}^*_{\bar{\Sigma}} = \dim \mathcal{S}^*_{\Sigma} = r$

Relating geometric subspaces of Σ and $\widehat{\Sigma}$

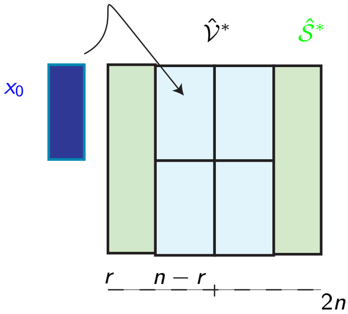


Denote by S and by $\bar{S}$ two basis matrices for $\mathcal{S}^*_{\Sigma}$ and $\mathcal{S}^*_{\bar{\Sigma}}$, respectively.

The largest input containing subspace $\mathcal{S}^*_{\widehat{\Sigma}}$ of $\widehat{\Sigma}$ has the following structure:

$$\mathcal{S}^*_{\widehat{\Sigma}} = \mathrm{im} \begin{bmatrix} S & \times \\ 0 & \bar{S} \end{bmatrix} \quad \text{and} \quad \dim \mathcal{S}^*_{\widehat{\Sigma}} = 2r$$

Main result



If (A, B, C) is left invertible, then $\widehat{\Sigma}$ is left and right invertible and $\mathcal{V}^*_{\widehat{\Sigma}} \oplus \mathcal{S}^*_{\widehat{\Sigma}} = \mathbb{R}^{2n}$. $\Longrightarrow$ being $\dim \mathcal{S}^*_{\widehat{\Sigma}} = 2r$, it follows that $\dim \mathcal{V}^*_{\widehat{\Sigma}} = 2(n-r)$.

- Modes in $\mathcal{V}^*_{\widehat{\Sigma}}$ are the zeros of $\widehat{\Sigma}$ that are all coupled by pairs $(z, -z)$.
- Modes in $\mathcal{V}^*_{\widehat{\Sigma}}$ are stable and unstable by pairs.

Recall: Statement of the problem - Cheap LQ problem

Consider the LTI system Σ

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t)\,, \quad x(0) = x_0 \\ y(t) &= C\,x(t) \end{aligned}$$

with the assumptions:

(A1) (A, B) is stabilizable

(A2) Σ has no invariant zeros on $j\,\mathbb{R}$

Determine the set of initial conditions x_0 such that a static state feedback matrix K exists s.t.

- $A - B\,K$ is stable
- the corresponding state trajectory minimizes

$$J(x, u) = \int_0^\infty y^\top(t)\,y(t)dt = \int_0^\infty x^\top(t) C^\top C\,x(t)\,dt$$

Recall: Disturbance decoupling

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C^{\mathsf{T}}C & -A^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t)$$

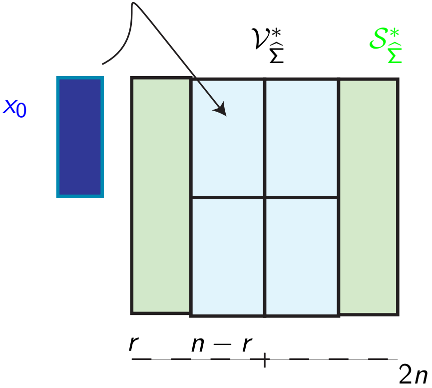
$$\tilde{y}(t) = \begin{bmatrix} 0 & B^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = 0$$

Problem: find $u(t) = K\,x(t)$ such that for $x(0) = x_0$

- $\tilde{y}(t) = 0$ for all $t \geq 0$
- $x(t) \to 0$ for $t \to \infty$.

This can be achieved if and only if x_0 is such that λ_0 exists so as that $\begin{bmatrix} x_0 \\ \lambda_0 \end{bmatrix}$ belongs to an internally stabilizable output-nulling subspace of $\widehat{\Sigma}$.

Resolvent subspace



If (A, B, C) is left invertible $\mathcal{V}^*_{\hat{\Sigma}} \oplus \mathcal{S}^*_{\hat{\Sigma}} = \mathbb{R}^{2n}$
Modes in $\mathcal{V}^*_{\hat{\Sigma}}$ are stable and unstable by pairs.
The dimension of $\mathcal{V}_R$ to which x_0 must belong for the solvability of the cheap problem is $n - r$.

The main results

- Let Σ be left invertible. An $(n-r)$-dimensional internally stabilizable output nulling subspace $\widehat{\mathcal{V}}_R$ of $\widehat{\Sigma}$ exists, such that all its $n-r$ poles, all unassignable, are stable.
- Let Σ be left invertible. The cheap problem is solvable if and only if $x_0 \in \mathcal{P}(\widehat{\mathcal{V}}_R)$ i.e., if and only if x_0 belongs to the projection of $\widehat{\mathcal{V}}_R$ on the state-space of Σ.

The static feedback

Let $\widehat{F}_R$ a friend of $\widehat{\mathcal{V}}_R$. Partition a basis matrix $\widehat{V}_R$ of $\widehat{\mathcal{V}}_R$ and $\widehat{F}_R$ as

$$\widehat{V}_R = \begin{bmatrix} V_X \\ V_\Lambda \end{bmatrix}, \quad \widehat{F}_R = \begin{bmatrix} F_X & F_\Lambda \end{bmatrix}$$

the state-feedback matrix K solving the problem is given by

$$K = -(F_X + F_\Lambda V_\Lambda V_X^+)$$

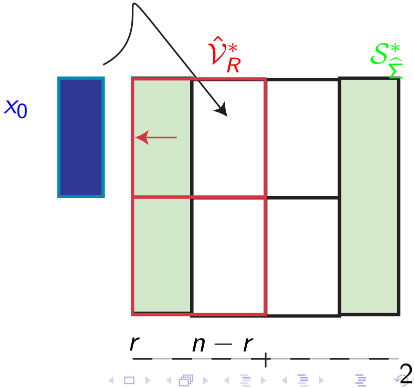
Cheap vs. regular problems

The dimension of $\mathcal{V}_R$ is $n - r$ where r is the dimension of $\mathcal{S}^*_\Sigma$.
Then

$$r \neq 0 \quad \text{and} \quad n - r < n$$

$\rightarrow$ there always exists $x_0 \notin \mathcal{V}_R$ for which the cheap LQ problem does not admit solutions with static feedback.

Regular problems. If the problem is regular, $\ker D = \{0\}$, then $\dim \mathcal{S}^* = 0$ and there always is a solution for any x_0.



Extension to non left invertible systems

Consider the auxiliary system $(A + B\,F, B\,U, C)$, where

1. F is such that $(A + B\,F)\,\mathcal{V}^*_\Sigma \subseteq \mathcal{V}^*_\Sigma$, and all the eigenvalues of $(A + B\,F)$ restricted to $\mathcal{V}^*_\Sigma \cap \mathcal{S}^*_\Sigma$ (which are free) are stable
2. U is a basis matrix of $(B^{-1}\,\mathcal{V}^*_\Sigma)^\perp$

The system thus obtained is left invertible.
Now, let K_F be the optimal state-feedback matrix for the auxiliary system.
Matrix

$$K := U\,K_F - F$$

is one of the **many** solutions of the original problem.
Different solutions correspond to different choices of matrix F.

Concluding remarks

- A new approach to the solution of the cheap linear quadratic regulator problem has been presented for continuous-time systems. This method is based on a geometric characterization of the structure of the Hamiltonian system.
- This insight is achieved by using the standard tools of the geometric approach, without resorting to changes of basis and reduced order algebraic Riccati equations.
- By recasting the cheap LQ problem as a perfect decoupling problem in the Hamiltonian system, we derive:
    1. the dimension of the subspace of the admissible initial conditions
    2. a basis matrix of this subspace, which is a controlled invariant solving a standard decoupling problem with stability for the Hamiltonian system
    3. an optimal state to input feedback matrix F, which is derived as a straightforward function of a friend of the above controlled invariant