



Spectral factorization and H_2 -model following

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Geometric Control Theory for Linear Systems

Block 1: Foundations [10:30 - 12.30]:

- Talk 1: *Motivation and historical perspective*, **G. Marro** [10:30 - 11:00]
- Talk 2: *Invariant subspaces*, **L. Ntogramatzidis** [11:00 - 11:30]
- Talk 3: *Controlled invariance and invariant zeros*, **D. Prattichizzo** [11:30 - 12:00]
- Talk 4: *Conditioned invariance and state observation*, **F. Morbidi** [12:00 - 12:30]

Block 2: Problems and applications [15:30 - 17.30]:

- Talk 5: *Stabilization and self-bounded subspaces*, **L. Ntogramatzidis** [15:30 - 16:00]
- Talk 6: *Disturbance decoupling problems*, **L. Ntogramatzidis** [16:00 - 16:30]
- Talk 7: *LQR and H_2 control and estimation problems*, **D. Prattichizzo** [16:30 - 17:00]
- Talk 8: *Spectral factorization and H_2 -model following*, **F. Morbidi** [17:00 - 17:30]

Introduction

The **spectral factorization problem** was introduced in the early '60s (Bauer and Youla), and it is at the core of modern system theory.

It arises in many fields:

- Wiener and Kalman filtering
- Stochastic realization (subspace identification)
- LQ-optimal control
- Network synthesis
- LQG game theory
- H_∞ control



D. C. Youla, "On the factorization of rational matrices", *IRE Trans. Information Theory*, vol. 7, no. 3, pp. 172–189, 1961.

Spectrum of a system

Consider the continuous-time LTI system Σ :

$$\begin{aligned}\dot{x}(t) &= A x(t) + B u(t), \\ y(t) &= C x(t)\end{aligned}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$.

Definition

The **spectrum** of Σ is defined as the function

$$\Phi(s) \triangleq G^*(s) G(s)$$

where

- $G(s)$ is the transfer function of Σ (the **spectral factor**)
- $G^*(s) \triangleq G^T(-s)$ is the transfer function of the *adjoint system*
 $\Sigma^* : (-A^T, -C^T, B^T)$

Cheap spectral factorization problem

Problem

Assume Σ is square, strictly stable, left and right invertible, observable and with no invariant zeros on the imaginary axis.

Determine a matrix C_1 such that the new system $\Sigma_1: (A, B, C_1)$, whose spectrum is $\Phi_1(s)$, satisfies the conditions:

- 1 $\Phi_1(s) = \Phi(s)$;
- 2 Σ_1 and Σ have the same number of outputs;
- 3 Σ_1 is minimum-phase (i.e., $\mathcal{Z}(\Sigma_1) \subseteq \mathbb{C}_g$);
- 4 Σ_1 has the same global relative degree ρ as Σ ;
- 5 Σ_1 has the same steady-state gain as Σ (i.e., $-C_1 A^{-1} B = -C A^{-1} B$).

Some remarks

- A *geometric solution* to the cheap spectral factorization problem has been recently proposed [1]
- It relies upon the solution to the **cheap LQ-optimal control problem** [2] (**Cheap** because the matrix R weighting the input of the cost J is zero)

Definition

Find the maximum set of initial states \mathcal{L} and a stabilizing state feedback control law $u(t) = -Kx(t)$ for Σ that minimizes the cost:

$$J = \frac{1}{2} \int_0^{\infty} x^T(t) C^T C x(t) dt$$

for any initial state $x_0 \in \mathcal{L}$.



G. Marro, F. Morbidi, D. Prattichizzo, "A Geometric Solution to the Cheap Spectral Factorization Problem", in *Proc. of European Contr. Conf.*, pp. 814–819, 2009.



D. Prattichizzo, L. Ntogramatzidis, G. Marro, "A new approach to the cheap LQ regulator exploiting the geometric properties of the Hamiltonian system," *Automatica*, vol. 44, no. 11, pp. 2834–2839, 2008.

Preliminaries

The **Hamiltonian system** $\widehat{\Sigma} : (\widehat{A}, \widehat{B}, \widehat{C})$ associated to $\Sigma : (A, B, C)$ is defined by

$$\begin{aligned}\dot{\widehat{x}}(t) &= \widehat{A} \widehat{x}(t) + \widehat{B} u(t), \\ 0 &= \widehat{C} \widehat{x}(t)\end{aligned}\quad \widehat{x} = \begin{bmatrix} x \\ p \end{bmatrix},$$

with

$$\widehat{A} = \begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} B \\ O \end{bmatrix},$$
$$\widehat{C} = [O \quad B^T].$$

Preliminaries

Properties of the Hamiltonian system $\widehat{\Sigma}$:

- 1 Σ is left-invertible $\implies \widehat{\Sigma}$ is both left and right-invertible.
- 2 Let Σ be left-invertible. The invariant zeros of $\widehat{\Sigma}$ are opposite by pairs. $\widehat{\Sigma}$ has no purely imaginary invariant zeros.
- 3 Let Σ be left-invertible. The invariant zeros of Σ are a subset of the invariant zeros of $\widehat{\Sigma}$, i.e.,

$$\mathcal{Z}(\Sigma) \subseteq \mathcal{Z}(\widehat{\Sigma}).$$

Let us introduce:

- $\mathcal{V}^* \triangleq \max \mathcal{V}(A, \text{im}B, \ker C), \quad \widehat{\mathcal{V}}^* \triangleq \max \mathcal{V}(\widehat{A}, \text{im}\widehat{B}, \ker \widehat{C}),$
- $\mathcal{S}^* \triangleq \min \mathcal{S}(A, \ker C, \text{im}B), \quad \widehat{\mathcal{S}}^* \triangleq \min \mathcal{S}(\widehat{A}, \ker \widehat{C}, \text{im}\widehat{B}).$

Preliminaries

Lemma

Let Σ be left-invertible and let $\dim \mathcal{S}^* = r$. Then, the following equalities hold:

$$\dim \widehat{\mathcal{S}}^* = 2r$$

$$\dim \widehat{\mathcal{V}}^* = 2(n - r)$$

Proposition

If Σ is left-invertible, an $(n - r)$ -dimensional internally stabilizable $(\widehat{A}, \text{im} \widehat{B})$ -**controlled invariant subspace** $\widehat{\mathcal{V}}_g^*$ **contained in** $\ker \widehat{C}$ exists, such that all its $n - r$ poles, all unassignable, are strictly stable.

- $\widehat{\mathcal{V}}_g^*$ is the **resolvent** controlled invariant subspace of $\widehat{\Sigma}$
- Let $\begin{bmatrix} V_o \\ P_o \end{bmatrix}$ be a basis matrix of $\widehat{\mathcal{V}}_g^*$ with $V_o, P_o \in \mathbb{R}^{n \times (n-r)}$
- Let $\mathcal{V}_o \triangleq \text{im } V_o$

Cheap spectral factorization: geometric solution

Theorem

There exists a unique matrix

$$C_1 \triangleq G \bar{G}^{-1} \bar{C}_1$$

where

- G is the steady-state gain of Σ ,
- \bar{G} is the steady-state gain of the triple (A, B, \bar{C}_1) ,
- \bar{C}_1 is the transpose of any basis matrix of the subspace

$$(\mathcal{V}_o + \mathcal{S}^* \cap \ker C)^\perp,$$

such that all the conditions of the cheap spectral factorization problem are met.

Some extensions

- The extension of the theorem to **non-right invertible** systems is immediate from an operative point of view:

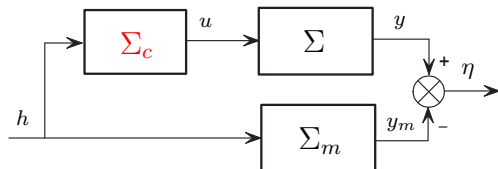
$$\boxed{C_1 \triangleq \mathbf{G} \bar{\mathbf{G}}^{-1} \bar{C}_1} \quad \mapsto \quad \boxed{C_1 \triangleq \mathbf{G} \bar{\mathbf{G}}^{\dagger} \bar{C}_1}$$

However, such an extension involves a number of delicate theoretical issues.

- It is **not** mandatory for the system Σ to be **strictly stable** and **observable**. Analogous results hold under *milder* assumptions.

Model following

Exact feedforward model following problem



Problem

Given a **plant** $\Sigma : (A, B, C)$ and a **model** Σ_m described by

$$\dot{x}_m(t) = A_m x_m(t) + B_m h(t)$$

$$y_m(t) = C_m x_m(t)$$

design a linear dynamic stable feedforward compensator $\Sigma_c : (A_c, B_c, C_c, D_c)$ such that the forced evolution of η is zero for every admissible (pw continuous) input h .

Model following

The exact feedforward model following problem is solvable [1] if:

$$\mathcal{Z}(\Sigma) \subseteq \mathbb{C}_g \quad (\text{stabilizability condition})$$

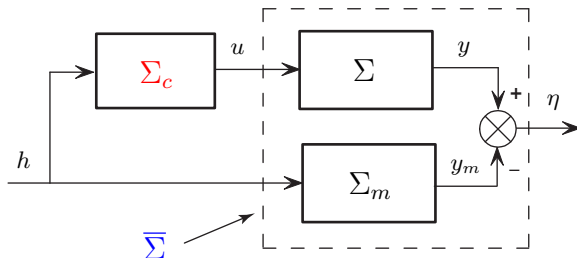
$$\rho(\Sigma) \leq \gamma(\Sigma_m) \quad (\text{structural condition})$$

where the **minimum delay** γ of $\Sigma_m : (A_m, B_m, C_m)$ is the minimum value of i such that $C_m A_m^i B_m$ is nonzero.



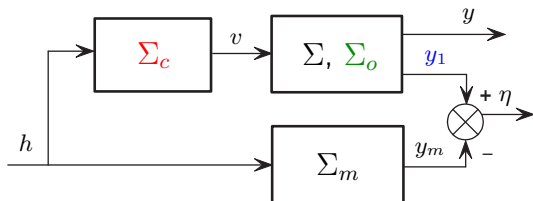
G. Marro, E. Zattoni, "Self-Bounded Controlled Invariant Subspaces in Model Following by Output Feedback: Minimal-Order Solution for Nonminimum-Phase Systems", *J. Optimiz. Theory Appl.*, vol. 125, no. 2, pp. 409–429, 2005.

Model following



The exact feedforward model following problem with stability is a particular case of the **exact measurable disturbance decoupling problem with stability** for an extended system $\bar{\Sigma}$ also including the model.

Spectral factorization and H_2 -model following



- When Σ is *nonminimum-phase*, a possible strategy consists in reformulating the model following problem in a **H_2 -minimal context**
- The output y_1 is provided through the matrix C_1 that solves the **cheap spectral factorization problem**
- The output y_1 may be provided by a standard **full or reduced order observer** Σ_o , that could also be used to provide a stabilizing feedback for Σ

Conclusion: H_2 -model following \rightarrow a special spectral factorization problem.

A numerical example

Let Σ be defined by

$$A = \begin{bmatrix} -1.19 & 16.32 & 9.43 & 10.83 \\ 2.15 & -0.35 & 0 & 0 \\ -8.84 & -34.16 & -14.49 & -6.33 \\ -4.53 & -16.15 & 0.39 & -5.56 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -1 \\ 0 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 6 \end{bmatrix}.$$

Σ is **nonminimum-phase**, in fact $\mathcal{Z}(\Sigma) = \{10.9786, -10.6053\}$.

Owing to the theorem, the output matrix,

$$C_1 = \begin{bmatrix} 2.5491 & 5.4594 & 4.2728 & 5.2290 \\ 0.5159 & -2.3140 & 3.7806 & 4.8226 \end{bmatrix}.$$

A numerical example

$\Sigma_1(A, B, C_1)$ satisfies the conditions of the *cheap spectral factorization problem*:

- 1 $\Phi_1(s) = \Phi(s)$;
- 2 C_1 and C have the same number of rows;
- 3 Σ_1 is minimum-phase being $\mathcal{Z}(\Sigma_1) = \{-10.9786, -10.6053\}$;
- 4 Σ_1 has the same global relative degree ($\rho = 1$) as Σ ;
- 5 Σ_1 and Σ have the same steady-state gain:

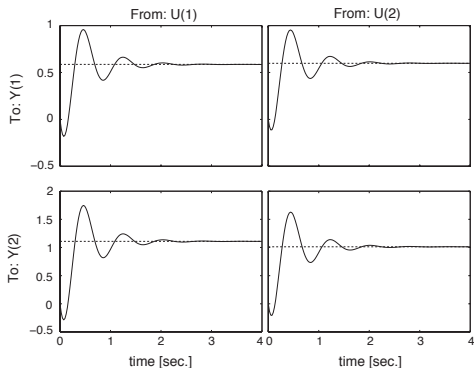
$$-C_1 A^{-1} B = -C A^{-1} B = \begin{bmatrix} 0.5862 & 0.5982 \\ 1.1090 & 1.0124 \end{bmatrix}.$$

Simulation results

- Matrix C_1 has been used to implement the *feedforward model following scheme*.
- The model Σ_m is defined by

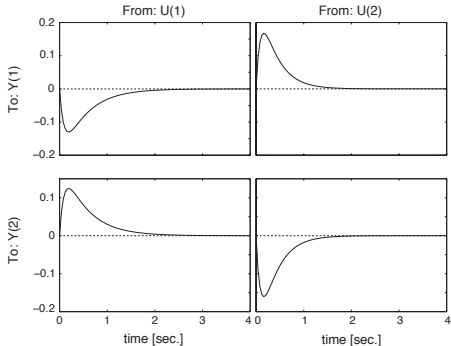
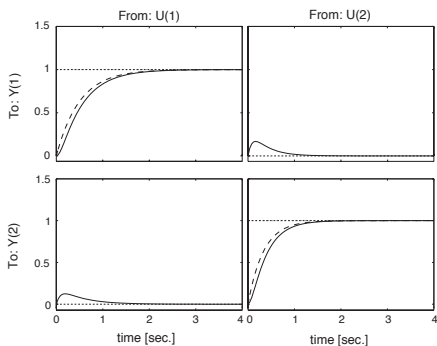
$$A_m = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}, \quad B_m = C_m = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix}.$$

- *Step response of Σ* :



Simulation results

- **Left:** *Step response* of the model Σ_m (---) and the regulated system $\Sigma_1\Sigma_c$ (—).
- **Right:** *Step tracking error*.



Computational tools

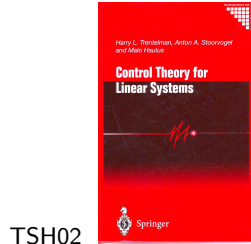
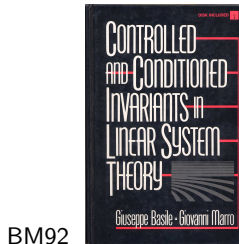
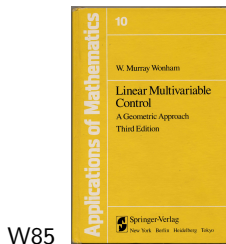
Matrix $C_1 = G\bar{G}^{-1}\bar{C}_1$ can be easily computed using the routines of the **Geometric Approach (GA) toolbox**:

<code>>> S = sstar(A,B,C,D);</code>	\mathcal{S}^*
<code>>> Vo = vstargh2(A,B,C,D);</code>	\mathcal{V}_o
<code>>> Vr = sums(Vo,ints(S,ker(C)));</code>	$\mathcal{V}_r = \mathcal{V}_o + \mathcal{S}^* \cap \ker C$
<code>>> C1bar = ortco(Vr)';</code>	$\bar{C}_1 = \text{transpose of any basis matrix of } \mathcal{V}_r^\perp$
<code>>> G = -C*inv(A)*B;</code>	$G = -CA^{-1}B$
<code>>> Gbar = -C1bar*inv(A)*B;</code>	$\bar{G} = -\bar{C}_1A^{-1}B$
<code>>> C1 = G*inv(Gbar)*C1bar;</code>	$C_1 = G\bar{G}^{-1}\bar{C}_1$

Summary

- Spectrum $\Phi(s)$ of an LTI system
- Cheap LQ-optimal control and cheap spectral factorization
- Properties of the Hamiltonian system $\hat{\Sigma}$
- Geometric solution to the cheap spectral factorization problem
- The H_2 -model following is a special spectral factorization problem

Topics **not** covered in this mini-course



- Regulator problem [W85, BM92, TSH02]
- H_∞ and robust control [TSH02, BM92]
- Pontryagin maximum principle [BM92]
- Fault detection and identification
- Geometric approach and nonlinear systems [Isidori, 1995]

Future challenges

- Model predictive control
- Control in the presence of input/state/output constraints
- Control of multidimensional systems
- Networked systems and distributed control
- J -spectral factorization
- Systems biology