Stable stochastic receding horizon control of linear systems with bounded control inputs

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Abstract—We address stability of receding horizon control for stochastic linear systems with additive noise and bounded control authority. We construct tractable and recursively feasible receding horizon control policies that ensure a mean-square bounded system in closed-loop if the noise has bounded fourth-order moment, the unexcited system is stabilizable, the system matrix \( A \) is Lyapunov stable, and there is large enough control authority.

I. INTRODUCTION

Receding horizon has become a natural way of dealing with the optimal control problem for linear systems involving input and state constraints [31], [29], [4], [15]. An inherent feature of receding-horizon methods is that the controllers may not be stabilizing in general. In the deterministic setting one starts with some initially feasible region and poses some terminal constraints which the state must satisfy at the end of the optimization horizon to ensure stability. In the stochastic setting involving possibly unbounded disturbances, however, no such terminal constraints can be posed. Moreover, imposing any bound on the control inputs with no restrictions on the system matrices may lead to the resulting system being unstable. At an intuitive level, this is quite natural since the control authority is restricted, while the disturbance can take large values, and can therefore drive the system unstable. In this article we aim to pose constraints under which receding horizon control policies can render the state of a given system mean-square bounded.

Receding horizon control of deterministic systems is a well-studied subject, see, e.g., [31], [4], [28], [15] and the references therein. In the stochastic setting, following early theoretical works on Markov decision processes [2], [21], [10], constructive methods with a strong applied flavor have received considerable attention. In [12], the authors first formulate the finite-horizon stochastic control problem as a deterministic one with bounded noise and then show that the underlying min-max problem is equivalent to a semidefinite program. In [36] a different problem is addressed in which the noise enters in a multiplicative manner, and hard constraints on the states and control inputs are relaxed to constraints resembling the integrated chance constraints of [20] or risk measures in mathematical finance. Another approach involving multiplicative noise and probabilistic input constraints has appeared in [16]. Similar relaxations of hard constraints to soft probabilistic ones have appeared in [33], [1] for additive noise inputs, as well as in [17] for both multiplicative and additive noise inputs. Alternative approaches employing randomized algorithms have been proposed in [39], [14], [3], [30]. Related lines of research can be found in [44], [45] dealing with constrained model predictive control (MPC) for stochastic linear systems motivated by industrial applications, in [41], [40] dealing with various tractable methods of controller synthesis, in [13] where optimality of a certain class of tractable feedback policies is established for deterministic scalar linear systems, and in [6] dealing with linear systems with multiplicative disturbances and using the scenario approach.

The deterministic version of the stabilization problem with bounded control inputs was solved completely in a sequence of articles [47], [43] culminating in [46]. It was demonstrated that global asymptotic stabilization of a discrete-time linear system \( x_{t+1} = Ax_t + Bu_t \) with bounded feedback controls is possible if and only if the system matrix \( A \) has spectral radius at most 1, and the pair \((A, B)\) is stabilizable with arbitrary controls. See also the recent work [25] dealing with input-to-state stability of bounded receding horizon controllers, and some earlier related work in [26]. In the context of stochastic linear systems \( x_{t+1} = Ax_t + Bu_t + w_t \) with \((w_t)_{t=0,1,2,...}\) being i.i.d noise vectors, it is possible to establish mean-square boundedness of the closed loop system under bounded receding horizon control policies by employing classical Foster-Lyapunov techniques [32] if the system matrix \( A \) is Schur stable [22], [18]. In this article we demonstrate that it is possible to strengthen this last result to the case of \( A \) being Lyapunov stable by considering the setup in the recent article by the authors [37]. It was shown in [37] that there exists a bounded control policy that ensures mean-square boundedness of the closed loop system if the pair \((A, B)\) is controllable, \( A \) is Lyapunov stable, and the process noise has bounded fourth moment. This is the track that we shall pursue. For related results see also [42].

The remainder of this article is organized as follows: We formulate the problem in Section II with all the underlying assumptions. In Section III we state the main result and prove it in Section IV. We conclude in Section V.

Notation

We denote by \( \mathbb{N} \) the set of non-negative integers. We denote by \( \|\cdot\| \) the standard Euclidean norm on \( \mathbb{R}^n \) and by \( \|\cdot\|_\infty \) the \( \ell_\infty \) norm. For any two matrices \( A \) and \( B \) of
compatible dimensions, we denote by \( R_k(A, B) \) the \( k \)-th step reachability matrix \( R_k(A, B) := [A^{k-1} B \cdots AB B] \).

For any matrix \( M \in \mathbb{R}^{n \times m} \), we let \( \sigma_1(M) \geq \cdots \geq \sigma_{\min(n,m)}(M) \) denote its singular values and \( M^\dagger \) denote its Moore-Penrose pseudo-inverse. We also denote by \((M)_{i:j}\) the matrix comprising the \( i \)-th to \( j \)-th rows of \( M \), where \( j \geq i \). For a \( n \times n \) positive semi-definite matrix \( M \), we let \( \|x\|_M := \sqrt{x^TMx} \) denote the weighted semi-norm of \( x \in \mathbb{R}^n \). Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a general probability space, we denote the conditional expectation given the sub-\( \sigma \) algebra \( \mathcal{G}' \) of \( \mathcal{G} \) as \( \mathbb{E}[\cdot | \mathcal{G}'] \). In a Euclidean space we denote by \( B_r \) the closed Euclidean ball or radius \( r \) centered at the origin. For \( r > 0 \) let the saturation function \( \text{sat}_r(y) = y \) if \( y \in B_r \), and \( \text{sat}_r(y) = ry/\|y\| \) otherwise. Note that \( \text{sat}_r(\cdot) \) is not the component-wise saturation function.

II. Problem Setup

A. Dynamics and Cost

Consider the following discrete-time linear system with additive noise

\[
x_{t+1} = Ax_t + Bu_t + w_t, \quad x_0 = x,
\]

where \( x_t \in \mathbb{R}^n, u_t \in \mathbb{R}^m, (w_t)_{t \in \mathbb{N}} \) is a sequence of \( n \)-dimensional random vectors, and \((A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}\).

We posit the following standing assumption:

**Assumption 1:**

1) The system matrices in (1) satisfy the following:
   a) \((A, B)\) is stabilizable [7, Chapter 12];
   b) \( A \) is discrete-time Lyapunov stable [7, Chapter 12], i.e., the eigenvalues \( \{\lambda_i(A) | i = 1, \ldots, d\} \) lie in the closed unit disc, and those eigenvalues \( \lambda_j(A) \) with \( |\lambda_j(A)| = 1 \) have equal algebraic and geometric multiplicities;
2) The noise sequence \((w_t)_{t \in \mathbb{N}} \) in (1) is a collection of independent \( n \)-dimensional random vectors having bounded fourth moment, i.e., \( C_4 > 0 \) such that \( \mathbb{E}[\|w_t\|^4] \leq C_4 < \infty \) for all \( t \in \mathbb{N} \).
3) The control inputs take values in the control constraint set

\[
U := \{ \xi \in \mathbb{R}^m \mid \|\xi\| \leq U_{\max} \},
\]

i.e., \( u_t \in U \) for all \( t \in \mathbb{N} \). ∎

Select \( C_1 > 0 \) such that \( \mathbb{E}[\|u_t\|] \leq C_1 \) for all \( t \in \mathbb{N} \); this is possible since Jensen’s inequality leads to \( \mathbb{E}[\|w_t\|] \leq \sqrt{\mathbb{E}[\|w_t\|^4]} \).

Without any loss of generality, we assume that \( A \) is in real Jordan canonical form. Indeed, given a linear system described by system matrices \((A, B)\), there exists a coordinate transformation in the state-space that brings the pair \((\tilde{A}, \tilde{B})\) to the pair \((A, B)\), where \( A \) is in real Jordan form [24, p. 150]. In particular, choosing a suitable ordering of the Jordan blocks, we can ensure that the pair \((A, B)\) has the form

\[
\begin{pmatrix}
A_s & 0 \\
0 & A_o
\end{pmatrix}
\begin{pmatrix}
B_s \\
B_o
\end{pmatrix},
\]

where \( A_s \in \mathbb{R}^{n_s \times n_s} \) is Schur stable, and \( A_o \in \mathbb{R}^{n_o \times n_o} \) has its eigenvalues on the unit circle. Due to the stability hypothesis Assumption 1-1b, \( A_o \) is therefore block-diagonal with elements on the diagonal being either \( \pm 1 \) or \( 2 \times 2 \) rotation matrices. As a consequence, \( A_o \) is orthogonal. Moreover, since \((A, B)\) is stabilizable, the pair \((A_s, B_s)\) must be reachable in a number of steps \( \kappa \leq n_o \) that depends on the dimension of \( A_s \) and the structure of \((A_s, B_s)\).

Summing up, we can start by considering that system (1) has the form

\[
\begin{bmatrix}
x^s_{t+1} \\
w^s_{t+1}
\end{bmatrix} = \begin{bmatrix}
A_s x^s_t \\
B_s y^s_t
\end{bmatrix} + \begin{bmatrix}
u^s_t \\
w^s_t
\end{bmatrix},
\]

(3)

where \( A_s \) is Schur stable and \( A_o \) is orthogonal. And, there exists a \( \kappa \leq n_o \) such that the subsystem \((A_o, B_o)\) is reachable in \( \kappa \) steps, i.e., \( \text{rank}(R_{\kappa}(A_o, B_o)) = n_o \). This integer \( \kappa \) is fixed throughout the rest of the article, and we will consider it to be our control horizon.

We fix a second horizon, namely an optimization (or prediction) horizon \( N \geq \kappa \). Given the state \( x_t \) at time \( t \), let us consider the following objective function:

\[
V_t := \mathbb{E}_{x_t}[\sum_{i=0}^{N-1} \left(\|x_{t+i}\|_{Q_i}^2 + \|u_{t+i}\|_{R_i}^2 + \|x_{t+N}\|_{Q_N}^2\right)],
\]

(4)

where \( Q_i \geq 0, R_i \geq 0, Q_N \geq 0 \) are given symmetrical matrices of appropriate dimensions. At times \( t = 0, \kappa, 2\kappa, \ldots \), we are interested in minimizing (4) over the class of causal \( N \)-history-dependent strategies \( \Pi_N \) defined as:

\[
\begin{bmatrix}
u_t \\
\vdots \\
u_{t+N-1}
\end{bmatrix} = \begin{bmatrix}
\pi_t(x_t) \\
\pi_{t+1}(x_{t+1}) \\
\vdots \\
\pi_{t+N-1}(x_t, x_{t+1}, \ldots, x_{t+N-1})
\end{bmatrix},
\]

(5)

for some measurable functions \( \{\pi_t, \ldots, \pi_{t+N-1}\} \in \Pi_N \), while satisfying the hard input constraints (2). The receding horizon control procedure with an optimization horizon \( N \) and a control horizon \( \kappa \) can be described as follows:

1) set \( t = 0 \);
2) measure the state \( x_t \);
3) determine an admissible optimal control policy \( \{\pi_t^*, \ldots, \pi_{t+N-1}^*\} \in \Pi_N \), that minimizes the \( N \)-stage cost function (4);
4) apply the first \( \kappa \) elements \( \{\pi_t^*, \ldots, \pi_{t+\kappa-1}^*\} \) of the policy;
5) increase \( t \) to \( t + \kappa \), and go back to step 2.

Therefore, the optimization problem to be solved at times \( t = 0, \kappa, 2\kappa, \ldots \) is given by:

\[
\min \{V_t \mid (1), (2), \{\pi_t, \ldots, \pi_{t+N-1}\} \in \Pi_N \}.
\]

(6)

The solution to Problem (6) is difficult to obtain in general. For instance, in order to obtain an optimal solution to Problem (6) over the class of causal state feedback policies, we need to solve the Dynamic Programming equations [9], [11]. This generally requires using gridding techniques, which suffer from the curse of dimensionality. Another approach is to restrict attention to a specific class of causal history-dependent policies. This will result in a suboptimal solution to our problem, but may yield a tractable optimization problem [8], [35]. It is this track that we pursue in the next section.
B. Policy Class

By the hypothesis that the state is fully observed without error, one may reconstruct the noise sequence from the sequence of observed states and inputs in Figure 1 by the formula

\[w_t = x_{t+1} - Ax_t - Bu_t, \quad t \in \mathbb{N}.\]  

(7)

![Block diagram of the closed-loop system.](image)

In light of this, we follow our earlier approach in [22], [18], [23] inspired by the works [38], [27], [5], [19] and use nonlinear N-history-dependent policies of the following form:

\[u_{t+l} = \eta_{t+l} + \sum_{i=0}^{l-1} \theta_{t+i, i} \varphi_i(w_{t+i+1}),\]  

(8)

for \(l = 0, 1, \ldots, N - 1\) and a measured state \(x_t\) at time \(t\), where \(\varphi_i(w_t)\) is any vector-valued function of \(w_t\) such that \(\|\varphi_i(w_t)\|_{\infty} \leq \varphi_{\text{max}}\) for all \(i\). In other words, we saturate the measurements that we obtain from the noise vectors before inserting them into our control vector. With this definition, the value of \(u\) at time \(t + l\) depends on the values of \(w\) from time \(t\) up to time \(t + l - 1\). Therefore, a finite amount of memory is required to compute the value of the inputs \(u_t\) for any \(t \in \mathbb{N}\). We do not assume that the noise distribution is defined over a compact domain, which is a crucial assumption in robust MPC approaches [12], [19]. Moreover, the choice of element-wise saturation functions \(\varphi_i(\cdot)\) is left open. As such, we can accommodate standard saturation, piecewise linear, and sigmoidal functions, to name a few.

Before we proceed to state the main result, it is helpful to have the following compact notation. The evolution of the saturation, piecewise linear, and sigmoidal functions, to name a few, can render the state of the closed-loop system mean-square bounded. Thus, a process noise does not have compact support. Instead, we introduce an additional stability constraint which, if feasible, can render the state of the closed-loop system mean-square bounded. Guided by our approach in [37] we then show that this constraint is indeed recursively feasible.

We require that the following stability constraint be satisfied: for any given \(\epsilon > 0\) and for every \(t = 0, \kappa, 2\kappa, \cdots, \) \(U_t \in U\) is chosen, such that the following “negative drift condition” holds:

\[
\begin{bmatrix}
0 & \cdots & 0 \\
I & \ddots & \\
A & \ddots & 0 \\
\vdots & \ddots & \ddots \\
A^{N-1}B & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
\begin{bmatrix} u_t \\ u_{t+N-1} \\ \vdots \\ u_{t+K-1} \\ \vdots \\ u_t + K \end{bmatrix} - \begin{bmatrix} x_t \\ x_t + \cdots \\ x_t + N-1 \end{bmatrix}
\end{bmatrix}
\leq - \left( \sqrt{K} \sigma_1 (R_t(A, I_\kappa)) C_1 + \frac{\epsilon}{2} \right),
\]

whenever \(\|x_t\| \geq \sqrt{K} \sigma_1 (R_t(A, I_\kappa)) C_1 + \epsilon\),

\[
\begin{aligned}
\|X_t^TQX_t + U_t^T R U_t\|
\end{aligned}
\]

where \(Q = \text{diag}(Q_0, Q_1, \cdots, Q_N)\) and \(R = \text{diag}(R_0, R_1, \cdots, R_{N-1})\). We can represent the control vectors over the optimization horizon as

\[
U_t = \eta_t + \Theta_t \varphi(W_t),
\]

(11)

where

\[
\begin{align*}
\eta_t & := \begin{bmatrix} \eta_t \\ \eta_{t+1} \\ \vdots \\ \eta_{t+N-1} \end{bmatrix}, \quad \varphi(W_t) := \begin{bmatrix} \varphi_0(w_t) \\ \varphi_1(w_{t+1}) \\ \vdots \\ \varphi_{N-2}(w_{t+N-2}) \end{bmatrix}, \\
\Theta_t & := \begin{bmatrix} 0 & 0 & \cdots & 0 \\
\theta_{t+1,t} & 0 & \cdots & 0 \\
\theta_{t+2,t} & \theta_{t+2,t+1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots \\
\theta_{t+N-1,t} & \theta_{t+N-1,t+1} & \cdots & \theta_{t+N-1,t+N-2} \end{bmatrix}.
\end{align*}
\]

(12)

Finally, the constraint (2) can be represented as

\[
U_t \in \{ \xi \in \mathbb{R}^{N} \mid \|\xi\|_\infty \leq U_{\text{max}} \}. 
\]

III. MAIN RESULT

It is well known that receding horizon control is in general not stabilizing, even in the noise-free case and without any bounds on the control inputs [27, p.26]. This problem is due to the fact that the finite-horizon optimization problem does not necessarily inherit the infinite-horizon stability property, usually present in the traditional infinite horizon Linear Quadratic (LQ) control problem. This problem is circumvented in deterministic MPC by requiring that the state at the end of the optimization horizon enters some terminal positively invariant set [31]. In our setup, however, no such constraint can be enforced on the state as the process noise does not have compact support. Instead, we introduce an additional stability constraint which, if feasible, can render the state of the closed-loop system mean-square bounded. Guided by our approach in [37] we then show that this constraint is indeed recursively feasible.
where $C_1$ is as defined earlier. The constraint (14) pertains only to the orthogonal subsystem in (3). Taking the policy structure in (11) and the stability constraint (14) into account, the optimization problem to be solved at times $t = 0, \kappa, 2\kappa, \cdots$ becomes:

$$
\min_{(\eta_i, \Theta_i)} \left\{ V_t \mid (9), (11), (13), \text{ and } (14) \right\}
$$

**Assumption 2:** In addition to Assumption 1, we stipulate that Assumption 2 holds. Then:

where $U$ satisfies. The matrices $\Omega$ space sequence of nonnegative random variables on some probability

Theorem 3 (Main Result): Consider system (1) and suppose that Assumption 2 holds. Then:

(i) For every $t = 0, \kappa, 2\kappa, \cdots$, the optimization problem (15) is convex, feasible, and can be rewritten as the following tractable program with tightened constraints:

$$
\min_{(\eta_i, \Theta_i)} \left\{ \text{tr}(\Theta_i^T \mathcal{M} \Theta_i, \Lambda^{p\varphi}) + 2 \text{tr}(\Theta_i^T B Q \Delta^{u\varphi}) + 2 R \eta^T \mathcal{W} \Theta_i \mathcal{M} \eta_i + \eta_i^T \mathcal{W} \mathcal{M} \Theta_i \Lambda^{p\varphi} \right\}
$$

subject to:

$$
\Theta_i \text{ having the structure in (12),}
$$

$$
\|A_o^\kappa x_o^\kappa + R_o(A_o, B_o)(\eta_i)_{1:x:n} \| + \sqrt{n_o} \|R_o(A_o, B_o)(\Theta_i)_{1:x:n} \| \varphi_{\max}
$$

where $\mathcal{M} = \mathcal{R} + B^T Q \mathcal{B}$, $\Lambda^{p\varphi} = E[\varphi(W_t)]$, $\Lambda^{u\varphi} = E[W_t^\kappa \varphi(W_t)^T]$, and $\mathcal{W}^{p\varphi} = E[\varphi(W_t)^T \varphi(W_t)^T]$. (ii) For any initial condition $x_0 \in \mathbb{R}^n$ successive application of the control laws given by the optimization problem in (i) for $k$ steps renders the closed-loop system mean-square bounded, i.e., there exists a constant $\gamma = \gamma(x_0, \epsilon, C_4, U_{\max})$ such that

$$
\sup_{t \in \mathbb{N}} \mathbb{E}_{x_0} \|x_t\|^2 \leq \gamma.
$$

A proof of Theorem 3 is provided in Section IV. Note that the constraint (18) is tightly equivalent to (13) (see [22]). The constraint (19) is a tightened representation of (14) for the control policy in (11), i.e., if (19) is satisfied then (14) is satisfied. The matrices $\Lambda^{p\varphi}$, $\Lambda^{u\varphi}$, and $\Lambda^{p\varphi}$ can be computed off-line as in [22], either in closed form or numerically, hence reducing the online computational burden.

IV. PROOF OF THEOREM 3

Let us first recall the following fundamental result.

**Proposition 4 ([34, Theorem 1]): Let $(\xi_t)_{t \in \mathbb{N}}$ be a sequence of nonnegative random variables on some probability space $(\Omega, \mathcal{F}, P)$, and let $(\xi_t)_{t \in \mathbb{N}}$ be any filtration to which $(\xi_t)_{t \in \mathbb{N}}$ is adapted. Suppose that there exist constants $\epsilon > 0, J < \infty$ and $M < \infty$, such that $\xi_0 \leq J$, and for all $t$, we have the following two conditions satisfied:

$$
\mathbb{E}[\xi_{t+1} - \xi_t] \leq \frac{-\epsilon}{2} \text{ on the event } \{\xi_t \geq J\},
$$

and

$$
\mathbb{E}[\xi_{t+1} - \xi_t \mid \xi_0, \ldots, \xi_t] \leq M.
$$

Then there exists a constant $\gamma = \gamma(\epsilon, J, M) > 0$ such that

$$
\sup_{t \in \mathbb{N}} \mathbb{E}[\xi_t^2] \leq \gamma.
$$

Proof of Theorem 3:

Proof of claim (i): The proof of convexity, the reformulation of the objective, and the constraints (17) and (18) can be found in [18], [22], and the constraint (19) is obviously convex as it is the sum of two norms. Therefore, it remains to show that the constraint (19) is both feasible and that it implies that the stability constraint (14) is satisfied.

Let us consider the feasibility of the constraint (19). Note that the matrix $R(A_o, B_o)$ has full rank equal to $n_o$. At any time $t = 0, \kappa, 2\kappa, \cdots$, consider the sequence of control input vectors

$$
\tilde{u}_{t,t+\kappa-1} := \begin{bmatrix} \tilde{u}_t \\ \tilde{u}_{t+1} \\ \vdots \\ \tilde{u}_{t+\kappa-1} \end{bmatrix} := -R(A_o, B_o)^\dagger \text{sat}_\epsilon(A_o^\kappa x_o^\kappa),
$$

for $r = (\sqrt{n_o}(R(A_o, I_{n_o}))C_1 + \frac{\epsilon}{2})$. Substituting the input above into the left-hand side of (14), we obtain

$$
\|A_o^\kappa x_o^\kappa - R(A_o, B_o)R(A_o, B_o)^\dagger \text{sat}_\epsilon(A_o^\kappa x_o^\kappa)\| - \|x_o^\kappa\| = = \|A_o^\kappa x_o^\kappa - \text{sat}_\epsilon(A_o^\kappa x_o^\kappa)\| - \|A_o^\kappa x_o^\kappa\|
$$

whenever $\|x_o^\kappa\| \geq (\sqrt{n_o}(R(A_o, I_{n_o}))C_1 + \frac{\epsilon}{2})$, and for all $r$:

$$
\|\tilde{u}_{t,t+\kappa-1}\|_\infty \leq \|\tilde{u}_{t,t+\kappa-1}\|_2
$$

$$
\leq \|R(A_o, B_o)^\dagger\|_2 \|\text{sat}_\epsilon(A_o^\kappa x_o^\kappa)\|_2
$$

whenever $\|x_o^\kappa\| \geq (\sqrt{n_o}(R(A_o, I_{n_o}))C_1 + \epsilon)$. Moreover, we have:

$$
\|\tilde{u}_{t,t+\kappa-1}\|_\infty \leq \|\tilde{u}_{t,t+\kappa-1}\|_2
$$

$$
\leq \|R(A_o, B_o)^\dagger\|_2 \|\text{sat}_\epsilon(A_o^\kappa x_o^\kappa)\|_2
$$

whenever $\|x_o^\kappa\| \geq (\sqrt{n_o}(R(A_o, I_{n_o}))C_1 + \epsilon)$. Moreover, this policy is encompassed by the general policy structure (11), as we can set the decision parameters to $(\bar{\eta}_t)_{1:x:n} = \tilde{u}_{t,t+\kappa-1}$ and $(\bar{\Theta}_t)_{1:x:n} = 0$.

Now, the sequence of control inputs can be written in terms of the decision variables as

$$
\tilde{u}_{t,t+\kappa-1} := \begin{bmatrix} \tilde{u}_t \\ \tilde{u}_{t+1} \\ \vdots \\ \tilde{u}_{t+\kappa-1} \end{bmatrix} = (\eta_t)_{1:x:n} + (\Theta_t)_{1:x:n} \varphi(W_t).
$$

(24)
Combining (24) with the stability constraint (14), we obtain
\[
\begin{align*}
\|A^\kappa_0 x^\kappa_t + R^\kappa(A_0, B_0) u_{t,t+t-\kappa-1}\| & = \|A^\kappa_0 x^\kappa_t + R^\kappa(A_0, B_0)(\eta^\kappa_1)_{1:k_m}^\kappa & + R^\kappa(A_0, B_0)(\Theta^\kappa)_{1:k_m}^\kappa \varphi(W_t)\| \\
& \leq \|A^\kappa_0 x^\kappa_t + R^\kappa(A_0, B_0)(\eta^\kappa_1)_{1:k_m}^\kappa & + R^\kappa(A_0, B_0)(\Theta^\kappa)_{1:k_m}^\kappa \varphi(W_t)\| \\
& \leq \|A^\kappa_0 x^\kappa_t + R^\kappa(A_0, B_0)(\eta^\kappa_1)_{1:k_m}^\kappa & + \sqrt{n_o} \|R^\kappa(A_0, B_0)(\Theta^\kappa)_{1:k_m}^\kappa \varphi(W_t)\|_{\infty} \\
& \leq \|A^\kappa_0 x^\kappa_t + R^\kappa(A_0, B_0)(\eta^\kappa_1)_{1:k_m}^\kappa & + \sqrt{n_o} \|R^\kappa(A_0, B_0)(\Theta^\kappa)_{1:k_m}^\kappa \varphi(W_t)\|_{\infty}.
\end{align*}
\]
Accordingly, if the constraint (19) is satisfied, then the stability constraint (14) is satisfied as well. This completes the proof of claim (i) of Theorem 3.

Proof of claim (ii): In order to show mean-square boundedness, we start by dividing the proof into two parts:
\[
\sup_{t \in \mathbb{N}} \mathbb{E}_{x_0} [\|x_t\|^2] = \sup_{t \in \mathbb{N}} (\mathbb{E}_{x_0} [\|x^\kappa_t\|^2] + \mathbb{E}_{x_0} [\|x^\kappa_0\|^2]) \\
\leq \sup_{t \in \mathbb{N}} \mathbb{E}_{x_0} [\|x^\kappa_t\|^2] + \sup_{t \in \mathbb{N}} \mathbb{E}_{x_0} [\|x^\kappa_0\|^2].
\]
It has been shown in [18] that for the Schur stable subsystem, under bounded control inputs and noise sequence with bounded second moment, there exists a constant \( \gamma_\kappa = \gamma_\kappa(x_0, C_4, U_{\text{max}}) > 0 \) such that
\[
\sup_{t \in \mathbb{N}} \mathbb{E}_{x_0} [\|x^\kappa_t\|^2] \leq \gamma_\kappa.
\] (25)

It remains to show that there exists a constant \( \gamma_\kappa = \gamma_\kappa(x_0, \epsilon, C_4, U_{\text{max}}) > 0 \) such that
\[
\sup_{t \in \mathbb{N}} \mathbb{E}_{x_0} [\|x^\kappa_0\|^2] \leq \gamma_\kappa.
\] (26)
We shall rely on the result in Proposition (4) and show that both conditions (21) and (22) are satisfied from which it follows that the orthogonal subsystem has bounded variance.

Consider the sub-sampled process \( x^\kappa_t \) \( t=0,1,2,\ldots \), given by
\[
x^\kappa_t = A^\kappa_0 x^\kappa_t + R^\kappa(A_0, B_0) u_{t,t+t-\kappa-1} + R^\kappa(A_0, I_{n_o}) w^\kappa_{t,t+t-\kappa-1},
\] (27)
where \( u_{t,t+t-\kappa-1} = \begin{bmatrix} u_t \\ \vdots \\ u_{t+t-\kappa-1} \end{bmatrix} \) and \( w^\kappa_{t,t+t-\kappa-1} = \begin{bmatrix} w^\kappa_t \\ \vdots \\ w^\kappa_{t+t-\kappa-1} \end{bmatrix} \).

Let \( \gamma_\kappa = \sigma(x_0, \cdots, x_\infty) \) and observe that
\[
\begin{align*}
\mathbb{E}^\kappa \left[ \|x^\kappa_{t+t-\kappa} - \|x^\kappa_t\| \right] & = \mathbb{E}^\kappa \left[ \left| A^\kappa_0 x^\kappa_t + R^\kappa(A_0, B_0) u_{t,t+t-\kappa-1} + R^\kappa(A_0, I_{n_o}) w^\kappa_{t,t+t-\kappa-1} \right| \right] \\
& \leq \mathbb{E}^\kappa \left[ \left| A^\kappa_0 x^\kappa_t + R^\kappa(A_0, I_{n_o}) w^\kappa_{t,t+t-\kappa-1} \right| \right] \\
& + \mathbb{E}^\kappa \left[ \left| R^\kappa(A_0, B_0) u_{t,t+t-\kappa-1} \right| \right]
\end{align*}
\]
Now using the stability condition (14), we can establish the upper bound:
\[
\begin{align*}
\mathbb{E}^\kappa \left[ \|x^\kappa_{t+t-\kappa} - \|x^\kappa_t\| \right] & \leq - \left( \sqrt{k} \sigma_1 (R_k(A_0, I_{n_o})) C_1 + \frac{\epsilon}{2} \right) \\
& + \mathbb{E} \left[ \| R^\kappa(A_0, I_{n_o}) w^\kappa_{t,t+t-\kappa-1} \| \right] \\
& \leq - \frac{\epsilon}{2},
\end{align*}
\]
for all \( x^\kappa_t \in \mathbb{R}^{n_o} \) such that \( \|x^\kappa_t\| > J := \sqrt{k} \sigma_1 (R_k(A_0, I_{n_o})) C_1 + \epsilon \), and condition (21) holds.

Also observe that
\[
\begin{align*}
\mathbb{E} \left[ \left( x^\kappa_{t+t-\kappa} - \|x^\kappa_t\| \right)^2 \right] & \leq \mathbb{E} \left[ \left( x^\kappa_{t+t-\kappa} - \|x^\kappa_t\| \right)^2 \right] \\
& + \mathbb{E} \left[ \left( x^\kappa_{t+t-\kappa} - \|x^\kappa_t\| \right)^2 \right] \\
& \leq \mathbb{E} \left[ \left( x^\kappa_{t+t-\kappa} - \|x^\kappa_t\| \right)^2 \right] \\
& \leq \mathbb{E} \left[ \left( x^\kappa_{t+t-\kappa} - \|x^\kappa_t\| \right)^2 \right] \\
& \leq \mathbb{E} \left[ \left( x^\kappa_{t+t-\kappa} - \|x^\kappa_t\| \right)^2 \right] \\
& \leq \mathbb{E} \left[ \left( x^\kappa_{t+t-\kappa} - \|x^\kappa_t\| \right)^2 \right]
\end{align*}
\] (28)

Since the control inputs are bounded by design and the forth moment of \( w_t \) is bounded by Assumption 1-2, expanding the last inequality in (28) and applying Jensen’s inequality shows that there exists a constant \( M = M(C_4, U_{\text{max}}) > 0 \) such that \( \mathbb{E} \left[ \left( x^\kappa_{t+t-\kappa} - \|x^\kappa_t\| \right)^2 \right] \leq M \). As such both hypotheses of Proposition (4) are satisfied with \( \tilde{\xi}_t := \|x^\kappa_t\| \), and hence the sub-sampled process \( x^\kappa_t \) \( t=0,1,2,\ldots \) is mean-square bounded, i.e., there exists a constant \( \gamma_\kappa = \gamma_\kappa(x_0, \epsilon, C_4, U_{\text{max}}) > 0 \) such that
\[
\sup_{t=0,1,2,\ldots} \mathbb{E}_{x_0} [\|x^\kappa_t\|^2] \leq \gamma_\kappa.
\] (29)

It follows from (29) via standard calculations involving triangle inequalities and the system dynamics that there exists another constant \( \gamma_\kappa = \gamma_\kappa(x_0, \epsilon, C_4, U_{\text{max}}) > 0 \) such that (26) holds. Finally, setting \( \gamma = \gamma_\kappa + \gamma_\kappa \) completes the proof of claim (ii) and hence the proof of Theorem 3.

V. CONCLUSIONS

We presented a method for stochastic receding horizon control of discrete-time linear systems with process noise and bounded control inputs. We showed that the optimization problem to be solved periodically is recursively feasible and convex under a suitable choice of control policies. Moreover, we showed how the optimization problem can be augmented with a certain stability constraint in order to render the state of the closed-loop system mean-square bounded. Future work will focus on extending this setup to the case of output feedback.

REFERENCES


