Observability Reduction of Piecewise-affine Hybrid Systems

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Abstract—We present necessary conditions for observability of piecewise-affine hybrid systems. We also propose an observability reduction algorithm for transforming a piecewise-affine hybrid system to a hybrid system of possibly smaller dimension which satisfies the formulated necessary condition for observability.

I. INTRODUCTION

In this paper we present necessary conditions for observability of piecewise-affine systems (abbreviated as PAHSs) and a dimensionality reduction procedure, based on this necessary condition.

A PAHS is a hybrid system, continuous dynamics of which is determined by affine control systems, the reset maps are affine and the guards are polyhedral sets. The definition of PAHS adopted in this paper is essentially the same as in [7], except that we allow the continuous state-space to be also a polyhedron as opposed to a polytope.

Contribution of the paper The contribution of the paper can be summarized as follows.

1. Necessary condition for observability We formulate an algebraic necessary condition for observability of a general PAHS. This condition is a generalization of the well-known rank condition for linear systems.

2. Linear PAHSs We introduce the class of linear PAHSs. A linear PAHS is a PAHS such that the control system in each discrete state is a linear (not affine) one, and lives on a polyhedron defined by linear inequalities, the reset maps are linear. For the class of linear PAHSs we propose necessary conditions, which are tighter than the ones for general PAHS. We refer to PAHSs which satisfy the latter condition as weakly observable ones. In particular, weak observability deals with discrete states with the same observable dynamics.

3. Observability reduction We formulate a procedure for transforming an arbitrary PAHS to a linear, weakly observable PAHS. The dimension of the thus obtained PAHS is bounded by the dimension of the original PAHS. Hence, the proposed transformation can be viewed as model reduction procedure aimed at merging observationally equivalent states.

Approach The main idea is to represent a (linear) PAHS $\Sigma$ as an output feedback interconnection of a linear hybrid system without guards (abbreviated as LHS, see [10], [9] for the definition) with an event generation device which detects crossing a guard, see Fig. 1. If we denote by $H$ the LHS above, and by $G$ the event generator, then $\Sigma$ is observable, only if $H$ is observable. Consider a LHS $H_o$ such that $H_o$ is observable and $H_o$ realizes the same input-output behavior as $H$. If we consider the interconnection of $H_o$ with the event generator $G$, we then obtain a PAHS $\Sigma_o$ with the following property. The LHS component of $\Sigma_o$ is observable and $\Sigma_o$ realizes the same input-output behavior as $\Sigma$.

Observability of LHSs is well-understood [10], [9], and the computation of $H_o$ can be carried out by an algorithm. Hence, the proposed necessary condition and observability reduction can be implemented numerically. In addition, the dimension of $H_o$ is not greater than that of $H$. In fact, we believe that this link between DPAHSs and LHSs is interesting on its own right and will be useful for problems other than observability analysis of PAHSs.

Related work There is a vast literature on observability of various classes of piecewise-linear hybrid systems, without claiming completeness, see [1], [2], [3], [5], [13], [11] and many others. However, most of the existing literature deals with observability of hybrid systems which are related but not identical to PAHSs. For example, typically reset maps are not considered, and the switching mechanism is assumed to be arbitrary, rather than state induced. The thus obtained conditions are either not directly applicable to PAHSs, or yield sufficient conditions. We believe that the necessary conditions of the paper represent a new result with respect to the existing literature.

To the best of our knowledge, observability reduction of PAHSs was addressed only in [8]. The current paper is an extension of [8].

Outline Section II presents the terminology and notation used in the paper. Section III defines piecewise-affine hybrid systems and the related system theoretic concepts. Section IV presents the main results of the paper. Section V presents a sketch of the proof of the main results.

II. PRELIMINARIES

Let $\Sigma$ be a finite set, referred to as the alphabet. $\Sigma^*$ denotes the set of finite strings (words) of elements of $\Sigma$, i.e. element of $\Sigma^*$ is a sequence $w = a_1a_2\cdots a_k$, where $a_1, a_2, \ldots, a_k \in \Sigma$, and $k \geq 0$; $k$ is the length of $w$ and it is denoted by $|w|$. If $k = 0$, then $w$ is the empty sequence (word), denoted by $\epsilon$. The concatenation of the words $v = v_1\cdots v_k$ and $w = w_1\cdots w_m \in \Sigma^*$ is the word $vw = v_1\cdots v_kw_1\cdots w_m$. The empty word $\epsilon$ is a unit element for concatenation. i.e. $\epsilon w = w \epsilon = w$ for all $w \in \Sigma^*$. 
Denote by \( \mathbb{N} \) the set of natural numbers including 0. Let \( T \) be the real time-axis, i.e. \( T = [0, +\infty) \). Denote by \( PC(T, \mathbb{R}^m) \) the set of piecewise-continuous maps (i.e. maps whose restriction to any finite interval is piecewise-continuous in the sense of \([6]\)) with values in \( \mathbb{R}^m \). For each \( n > 0 \) and \( j = 1, 2, \ldots, n \), \( e_j \) is the \( j \)th standard unit basis vector of \( \mathbb{R}^n \), i.e. \( e_j = (\sigma_{1,j}, \sigma_{2,j}, \ldots, \sigma_{n,j})^T \), where \( \sigma_{j,j} = 1 \) and \( \sigma_{i,j} = 0 \) for \( i \neq j \).

### III. PIECEWISE-AFFINE HYBRID SYSTEMS

**Definition 1:** A continuous-time piecewise-affine hybrid system (abbreviated as \( \text{PAHS} \)) is a hybrid system in the sense \([12]\) of the following form

\[
\begin{align*}
\dot{x}(t) &= A_q(t)x(t) + B_q(t)u(t) + a_q(t) \\
y(t) &= C_q(t)x(t) + c_q(t) \\
o(t) &= \delta(q(t), \gamma(t)) \quad \text{and} \quad q(t^+) = \delta(q(t), \gamma(t)) \\
x(t^+) &= M_q(t^+), \gamma(t), q(t^+) + m_q(t^+), \gamma(t), q(t) \\
\gamma(t) &= e \iff n_{q(t),e}^T x(t) = b_q, \quad \text{and} \quad n_{q(t),e}^T x(t) > 0
\end{align*}
\]

(1)

The various parameters are as follows

- \( q(t) \in Q \) is the discrete state at time \( t \), and \( Q \) is the finite set of discrete states (modes),
- \( o(t) \in O \) is the discrete output at time \( t \), and \( O \) is the finite set of discrete outputs,
- \( \gamma(t) \in \Gamma \) is discrete event at time \( t \), and \( \Gamma = \{1, 2, \ldots, E\} \), \( E > 0 \) is the finite set of discrete events,
- \( \delta : Q \times \Gamma \rightarrow Q \) is the discrete state-transition map,
- \( \lambda : Q \rightarrow O \) is the discrete readout map,
- For each \( q \in Q \), the affine system is described by matrices \( A_q \in \mathbb{R}^{n_x \times n_x} \), \( B_q \in \mathbb{R}^{n_y \times m} \), \( C_q \in \mathbb{R}^{p \times n_y} \)
- The state \( x(t) \) of \( \Sigma \) associated with the discrete state \( q \in Q \) lives on the (convex) polyhedron \( \mathcal{P}_q \) of the form

\[
\mathcal{P}_q = \bigcap_{\gamma \in \Gamma} \{ x \in \mathbb{R}^{n_x} | n_{q,\gamma}^T x \leq b_q, \gamma \}
\]

where \( n_{q,i} \in \mathbb{R}^{n_x} \), \( b_q \in \mathbb{R} \). The facets of \( \mathcal{P}_q \) are called exit facets of the polyhedron \( \mathcal{P}_q \).

- \( x(t) \in \mathbb{R}^{n_x(t)} = X_q(t) \) is the continuous state at time \( t \),
- \( y(t) \in \mathbb{R}^p \), for \( p > 0 \), is the continuous output at time \( t \), and \( \mathbb{R}^p \) is the space of continuous outputs,
- \( u(t) \in \mathbb{R}^m, m > 0 \), is the continuous input at time \( t \), and \( \mathbb{R}^m \) is the space of continuous inputs,
- The transition between the different continuous state-spaces takes place via affine reset map \( R_{q+, \gamma}, q \in Q, \gamma \in \Gamma \), \( \dot{q} = \delta(q, \gamma) \) where \( R_{q+, \gamma}, q \in Q \), \( \gamma \in \Gamma \), \( q^+ = M_{q+, \gamma}, q \in \mathbb{R}^{n_x \times n_x} \), and \( m_{q+, \gamma} \in \mathbb{R}^{n_y \times n_x} \).
- \( h_0 = (q_0, x_0) \), \( x_0 \in \mathcal{P}_{q_0} \) is the initial state of \( \Sigma \).

The state space \( \mathcal{H}_\Sigma \) of \( \Sigma \) is \( \mathcal{H}_\Sigma = \bigcup_{q \in Q} \{q\} \times \mathcal{P}_q \).

For the definition of evolution of a \( \text{PAHS} \) see \([7]\). Note that in contrast to \([7]\), we also allow discrete outputs. In addition, we do not require the sets \( \mathcal{P}_q \) to be polytopes, but only polyhedrons. However, the case of \( \mathcal{P}_q \) to be a polytope is a special case of the above definition. If \( \Sigma \) is a \( \text{PAHS} \) such that for each \( q \in Q \), the polyhedron \( \mathcal{P}_q \) is also a polytope, then we say that \( \Sigma \) is an \( \text{PAHS on polytopes} \).

The evolution of \( \Sigma \) takes place according to the definition \([12], [7]\). Assume that we feed in a \( \mathbb{R}^m \)-valued input signal \( u(t) \in \mathbb{R}^m \). As long as the value of the discrete state \( q \) does not change, the continuous state and the continuous output change according to the affine system \( \dot{x}(t) = A_q(t)x(t) + B_q(t)u(t) + a_q(t) \) and \( y(t) = C_q(t)x(t) + c_q(t) \). The discrete state changes only when a discrete event occurs. A discrete event \( e \in \Gamma \) takes place if the continuous state \( x(t) \) is about to leave the polyhedron \( \mathcal{P}_q \) through the exit facet associated with \( \gamma \), i.e. \( n_{q,\gamma}^T x(t^+) = b_q, \gamma \) and \( n_{q,\gamma}^T x(t^+) = n_{q,\gamma}^T (A_q x(t^+) + B_q u(t) + a_q) > 0 \). Here \( x(t^+) \) is the state just before the discrete event occurs, i.e. \( x(t^+) \) is the left-hand side limit \( x(t^-) = \lim_{t \to t^-} x(s) \). Then the new discrete state is determined by the discrete state-transition rule as \( q^+ = \delta(q, \gamma) \). The new continuous state \( x(t) = x(t^+) \in \mathbb{R}^{n_x(t^+)} \) is obtained by applying the corresponding reset map, that is, \( x(t^+) = M_{q+, \gamma, q} x(t^-) + m_{q+, \gamma, q} \). The discrete output is obtained from the discrete state by applying the discrete readout map, i.e. \( o = \lambda(q) \). After that, the continuous state and output evolve according to the affine system associated with the new discrete state \( q^+ \).

Note that the event generation mechanism described above is non-deterministic, since the continuous state might cross several exit facets at the same time. In turn, this non-determinism could lead to non-uniqueness of state and output trajectory. In order to ensure existence of a unique state and output trajectory, we will parameterize \( \text{PAHSs} \) by so called event generators, i.e. maps which choose which of the potentially enabled events should be generated.

**Definition 2 (Event-generator):** An even generator is a partial map

\[
G : \mathbb{R}^E \times \mathbb{R}^E \rightarrow \Gamma
\]

such that for any \( h = (z, \dot{z}) \) the following holds.

1. If \( G(h) \) is defined and \( G(h) = e \), then \( z_e = 0 \) and \( \dot{z}_e > 0 \). Here \( z_e \) and \( \dot{z}_e \) denote the \( e \)th entry of \( z \) and \( \dot{z} \) respectively.
2. If \( G(h) \) is not defined then for all \( e \in \Gamma \), either \( z_e \neq 0 \) or \( \dot{z}_e \leq 0 \).

In order to use even generators to describe the behavior of a \( \text{PAHS} \) \( \Sigma \), it is useful to introduce the following notation.

**Definition 3 (Guard map):** Assume that \( \Sigma \) is a \( \text{PAHS} \) of the form (1). The guard map of \( \Sigma \) is a map

\[
G_{\Sigma} : \bigcup_{q \in Q} \{q\} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \rightarrow \mathbb{R}^E \times \mathbb{R}^E
\]

defined as follows. For any \((q, x, \dot{x})\),

\[
G_{\Sigma}(q, x, \dot{x}) = \left( \begin{array}{c}
\left[ n_{q,1}^T x - b_{q,1} \\
n_{q,2}^T x - b_{q,2} \\
\vdots \\
n_{q,E}^T x - b_{q,E}
\end{array} \right]
\end{array} \right).
\]

The intuition behind the definition is as follows. \( G_{\Sigma} \) maps a hybrid state and the derivative of the continuous part to their scalar product with the normal vectors of the exit facet.
of the polyhedron. More precisely, if the current state is
$h = (q, x)$, then $G_x(q, x, \dot{x})$ contains all the information
the event generator $G$ needs in order to decide which event
to generate. That is, an event $e$ is generated by $\Sigma$ under
the event generator $G$, if $G(G_x(q, x, \dot{x})) = e$ and no event is
generated if $G(G_x(q, x, \dot{x}))$ is undefined.

Condition 1 of Definition 2 says that if $G$ generates and
event $e$, then this can happen only if the continuous state is
about to cross one of the facets of the polyhedron. Condition
2 requires that $G$ does not miss an event, i.e. if $G(h)$ is not
defined, then the current continuous state is either inside the
polyhedron or it is sliding along one of the facets.

Equipped with the notion of an event-generator, we can
define a unique state trajectory of the PAHS $\Sigma$ for each event
generator. To this end, we introduce the following definition.

Definition 4: A determined PAHS (abbreviated as
DPAHS ) is a pair $(\Sigma, G)$, where $\Sigma$ is a PAHS of the form
$(1)$ and $G$ is an event generator.

Definition 5 (Time event sequence, [4]): A time event
sequence is a strictly monotone sequence $\{t_n\}_{n=0}$ such that
$n^* \in \mathbb{N} \cup \{+\infty\}$, $t_0 = 0$ and for all $0 < n < n^*$,
$0 \leq t_n < t_{n+1}$. If $n^* = +\infty$ then let $t_\infty = \operatorname{sup}\{t_n | n \in \mathbb{N}\}$.

Notation 1: For a set $A$, $A^T$ is the set of all maps $f : [0, T_f) \to A$, where $T_f \in T \cup \{+\infty\}$.

Definition 6 (Input-to-state map): Consider a DPAHS
$(\Sigma, G)$, assume that $\Sigma$ is of the form (1). Recall that $\mathcal{H}_\Sigma$
is the state-space of $\Sigma$. For any state $h = (q_{init}, x_{init})$, with
$q_{init} \in Q$, $x_{init} \in \mathcal{P}_{\Sigma_{init}}$, define the input-to-state map of
$(\Sigma, G)$ induced by the state $h$ as

$$x_{\Sigma,G,h} : PC(T, \mathbb{R}^m) \to \mathcal{H}_\Sigma^T$$

such that for any $u \in PC(T, \mathbb{R}^m)$, $x_{\Sigma,G,h}(u) : [0, T_{f,u}) \to \mathcal{H}_\Sigma$, where $T_{f,u}$ depends on $h$ and
the following holds. There exists a time event sequence $(t_n)_{n=0}^{n^*}$ such that

$$T_{f,u} = \left\{ \begin{array}{ll}
+\infty & \text{if } n^* < +\infty \\
t_\infty & \text{if } n^* = +\infty
\end{array} \right.,$$

and there exists a (possibly empty) sequence $(\gamma_k)_{k=1}^{n^*}$ of
events from $\Gamma$, such that the following holds.

1) $x_{\Sigma,G,h}(u)(0) = h$

2) For all $i \in \mathbb{N}$, $i \leq n^*$, and all $t \in [0, T_{f,u})$ such that
t $t \in [t_i, t_{i+1})$, for $i = n^* < +\infty$, $t_{n^*+1} = +\infty$,
$$x_{\Sigma,G,h}(u)(t) = (q(i), x_i(t))$$

where $q(i) \in Q$ and the map $x_i : [t_i, t_{i+1}) \to \mathcal{P}_{q(i)}$
satisfies the differential equation

$$\dot{x}_i(t) = A_{q(i)}x_i(t) + B_{q(i)}u(t) + a_{q(i)},$$

and for all $t \in [t_i, t_{i+1})$, $G(G_x(q(i), x_i(t), \dot{x}_i(t)))$
is undefined.

3) For all $i \in \mathbb{N}$, $i < n^*$, the following holds. If $t_{i+1} = 0$, then
$(q(0), x(0)) = h$, and if $t_{i+1} > 0$, then
$$\forall t \in [t_i, t_{i+1}) : x_{\Sigma,G,h}(u)(t) = (q(i), x_i(t))$$
$$x_i(t_{i+1}) = \lim_{t \to t_{i+1}^-} x_i(t)$$

Then,

$$G(G_x(q(i), x_i(t_{i+1}), \dot{x}_i(t_{i+1}))) = \gamma_{i+1} \in \Gamma$$

$$\dot{x}_i(t_{i+1}) = A_{q(i)}x_i(t_{i+1}) + B_{q(i)}u(t_{i+1}) + a_{q(i)}$$

$$x_{\Sigma,G,h}(u)(t_{i+1}) = (q(i+1), x_{i+1}(t_{i+1}))$$

$q(i+1) = \delta(q(i), \gamma_{i+1})$

$$x(t_{i+1}) = M_{q(i+1), \gamma_{i+1}, q(i)}x_i(t_{i+1}) + m_{q(i+1), \gamma_{i+1}, q(i)}$$

Remark 1 (Well-posedness of $x_{\Sigma,G,h}$): Note that $x_{\Sigma,G,h}$
need not be exist for all states $h$. Intuitively, $x_{\Sigma,G,h}$, exists,
if no state trajectory starting from $h$ allows generation of
several consecutive events, such that one events occurs
immediately after another one. Note that in practice for most
of systems the latter scenario will not occur, and hence
$x_{\Sigma,G,h}$ will exist. If each reset map $R_{q^*, \gamma, q}$ of $\Sigma$ maps
the boundary of the polyhedron $\mathcal{P}$ into the interior of the
polyhedron $\mathcal{P}^+$, then it is easy to see that $x_{\Sigma,G,h}$, exists for
any state $h$ of $\Sigma$. Note that if $x_{\Sigma,G,h}$, exists, then it is unique.

Remark 2: In the definition above, $n^* = +\infty$ and $t_\infty < +\infty$
corresponds to Zeno-behavior.

Assumption 1: In the sequel, for any PAHS $\Sigma$ considered,
it is assumed that for every event generator $G$, the input-to-
state map $x_{\Sigma,G,h_0}$, induced by the initial state $h_0$ of $\Sigma$, exists.

Definition 7 (Input-output map): Assume that $h$ is a state of
the DPAHS $(\Sigma, G)$, such that the map $x_{\Sigma,G,h}$ exists. Then
the input-output map $y_{\Sigma,G,h} : (\Sigma, G)$ induced by $h$ is a map

$$y_{\Sigma,G,h} : PC(T, \mathbb{R}^m) \to (O \times \mathbb{R}^p)^T$$

such that for all $u \in PC(T, \mathbb{R}^m)$ the domain of $y_{\Sigma,G,h}$ is
the same as that of $x_{\Sigma,G,h}$, i.e. it is $[0, T_{f,u})$, and for every
t $t \in [0, T_{h,u})$, if $x_{\Sigma,G,h}(u)(t) = (q, x)$, then

$$y_{\Sigma,G,h}(u)(t) = (\lambda(q), C_{q}x)$$

Note that $y_{\Sigma,G,h}$ need not exist for all states $h$ of $\Sigma$; the
input-output map $y_{\Sigma,G,h}$ exists precisely when the input-to-
state map $x_{\Sigma,G,h}$ exists. In particular, due to the Assumption
1, $y_{\Sigma,G,h_0}$ exists, where $h_0$ is the initial state of $\Sigma$.

Definition 8 (Observability): Two states $h_1$ and $h_2$ of a
DPAHS $(\Sigma, G)$ are indistinguishable, if the input-outputs
maps $y_{\Sigma,G,h_1}$ and $y_{\Sigma,G,h_2}$ exist and they are equal, i.e. if
$y_{\Sigma,G,h_1} = y_{\Sigma,G,h_2}$. Note that the equality of $y_{\Sigma,G,h_1}$
$y_{\Sigma,G,h_2}$ also implies that the domains of $y_{\Sigma,G,h_1}(u)$ and
$y_{\Sigma,G,h_2}(u)$ are the same for all $u \in PC(T, \mathbb{R}^m)$. The
DPAHS $(\Sigma, G)$ is called observable, if there exists no pair of
distinct indistinguishable states, i.e. if for any $h_1, h_2 \in \mathcal{H}_\Sigma$
such that $y_{\Sigma,G,h_1}$ and $y_{\Sigma,G,h_2}$ exist, $y_{\Sigma,G,h_1} = y_{\Sigma,G,h_2}$
implies $h_1 = h_2$. The PAHS $\Sigma$ is observable, if for any event
generator $G$, the DPAHS $(\Sigma, G)$ is observable.
Note that observability has implications only for those states,
for which the input-output map exists.

Definition 9 (Realization): The input-output map of a
DPAHS $(\Sigma, G)$, denoted by $y_{\Sigma,G}$, is the input-output map of
$(\Sigma, G)$ induced by the initial state $h_0$ of $\Sigma$, i.e. $y_{\Sigma,G} = y_{\Sigma,G,h_0}$. An input-output map $f : PC(T, \mathbb{R}^m) \to (O \times \mathbb{R}^p)^T$
is said to be realized by $(\Sigma, G)$, if $f = y_{\Sigma,G}$.

Recall that the dimension of a polyhedron $\mathcal{P}$ equals $n$, if
there exists $n+1$ affinely independent elements of $\mathcal{P}$ which
constitute an affine basis of the affine hull of $\mathcal{P}$.
Definition 10 (Dimension): For each discrete state \( q \in Q \), denote by \( d_q \) the dimension of \( \mathcal{P}_q \). Then the dimension of the PAHS \( \Sigma \), denoted by \( \dim \Sigma \), equals the pair \((|Q|, \sum_{q \in Q} d_q)\).

In the sequel, we will use the following ordering on pairs of natural numbers \((m, n) \leq (p, q)\), if \( m \leq p \) and \( n \leq q \). That is, the pair \((m, n)\) is smaller than or equal to the pair \((p, q)\), if \( m \) is not greater than \( p \) and \( n \) is not greater than \( q \). Notice that the above ordering is a partial order. That is, there can be two PAHSs with incomparable dimensions.

IV. MAIN RESULTS

Below we present the main results of the paper. Throughout this section, \( \Sigma \) denotes a PAHS of the form (1).

### Notation 2 (Augmented output matrices): Recall that \( \Gamma = \{1, 2, \ldots, E\} \). For any discrete state \( q \in Q \),

\[
\bar{C}_q = \begin{bmatrix} C_q & n_{q,1} & n_{q,2} & \ldots & n_{q,E} \end{bmatrix}^T \in \mathbb{R}^{P \times E} \tag{2}
\]

In other words, \( \bar{C}_q \) is a block matrix, obtained by vertically ‘stacking up’ the matrix \( C_q \) and the normal vectors of the facets of \( \mathcal{P}_q \). The matrix \( \bar{C}_q \) corresponds to the readout matrix of the system \( \Sigma \) associated with \( \Sigma \).

### Notation 3 (Augmented event set): Let \( e \) be a symbol not in \( \Gamma \), and define the set \( \bar{\Gamma} = \Gamma \cup \{e\} \).

### Notation 4 (Discrete state-transition map): We extend to discrete-state transition map \( \bar{\delta} \) to a map \( \bar{\delta} : Q \times \bar{\Gamma}^* \rightarrow Q \) as follows. For any discrete state \( q \in Q \) and sequence \( w \in \bar{\Gamma}^* \), define the discrete state \( \bar{\delta}(q, w) \) recursively as follows.

- If \( w = e \), then \( \bar{\delta}(q, w) = q \).
- If \( w = \sigma \nu \) for some \( \sigma \in \bar{\Gamma} \), \( \nu \in \bar{\Gamma}^* \), then

\[
\bar{\delta}(q, w) = \begin{cases} 
\bar{\delta}(q, \nu) & \text{if } \sigma = e \\
\delta(\bar{\delta}(q, \nu), \gamma) & \text{if } \sigma = \gamma \in \Gamma
\end{cases}
\]

By abuse of notation, we denote the extension \( \bar{\delta} \) of the discrete state-transition map by \( \delta \) as well.

### Notation 5 (Product of matrices): For any \( q \in Q \) and sequence \( w \in \bar{\Gamma}^* \) define the \( n_q \times n_q \) matrix \( \Pi(q, w) \), where \( \bar{q} = \delta(q, w) \), recursively as follows.

- If \( w = e \), then \( \Pi(q, w) = I_{n_q} \), where \( I_{n_q} \) is the \( n_q \times n_q \) identity matrix.
- If \( w = \nu \sigma \) for some \( \sigma \in \bar{\Gamma} \), \( \nu \in \bar{\Gamma}^* \), then

\[
\Pi(q, w) = \begin{cases} 
A_q \Pi(q, \nu) & \text{if } \sigma = e \\
M_{\delta(q, \nu), \gamma, q} \Pi(q, \nu) & \text{if } \sigma = \gamma \in \Gamma
\end{cases}
\]

Define the \( pE \times n_q \) output matrix \( \mathcal{O}(q, w) \) as follows

\[
\mathcal{O}(q, w) = \hat{C}_{\delta(q, w)} \Pi(q, w)
\]

Next, we introduce the generalization of the notion of Markov parameters for PAHSs.

### Definition 11 (Markov parameters): The Markov parameter of \( \Sigma \) indexed by discrete state \( q \in Q \), sequences \( w \in \bar{\Gamma}^* \) and \( v \in \bar{\Gamma}^* \) is defined as the following matrix

\[
\mathcal{M}_q(w, v) = \mathcal{O}(q, w) B_{\delta(q, v)} \in \mathbb{R}^{P \times E \times m}
\]

Note that in the definition of the markov parameter \( \mathcal{M}_q(w, v) \), the sequence \( v \) is composed only of discrete events, while \( w \) can contain the addition symbol \( e \). The above definition is inspired by theory of LHSs, in fact the Markov-parameters of \( \Sigma \) are the Markov-parameters of the LHS \( \Sigma \). Intuitively, the Markov parameter \( \mathcal{M}_q(w, v) \) corresponds to a certain derivative of the continuous output generated from the discrete state \( \delta(q, v) \), with respect to the event times.

Finally, for any discrete state \( q \in Q \) of \( \Sigma \) we define the generalization of observability subspace.

### Definition 12 (Observability kernel): For any discrete state \( q \in Q \) of \( \Sigma \) define the observability subspace \( \mathcal{O}_{\Sigma, q} \) of \( \Sigma \) as a subset of \( \mathbb{R}^{n_q} \) of the following form

\[
\mathcal{O}_{\Sigma, q} = \bigcap_{w \in \bar{\Gamma}^*} \ker \mathcal{O}(q, w)
\]

The space \( \mathcal{O}_{\Sigma, q} \) is a generalization of the observability space for linear systems. In fact, \( \mathcal{O}_{\Sigma, q} \) is contained in the observability space of the linear system \( \{A_q, C_q\} \). However, \( \mathcal{O}_{\Sigma, q} \) also takes into account the output after first, second, etc. discrete-state transition. That is why products of the matrices of the affine subsystems and of the reset maps are considered too. The space \( \mathcal{O}_{\Sigma, q} \) is identical to the observability space \( \mathcal{O}_{\Sigma, q} \) of \( \Sigma \) associated with \( \Sigma \).

### Definition 13 (Full-dimensional PAHS): A PAHS \( \Sigma \) of the form (1) is called full-dimensional, if for any \( q \in Q \), the dimension of the polyhedron \( \mathcal{P}_q \) equals \( n_q \).

### Definition 14 (Complete PAHS): We say that a PAHS \( \Sigma \) is complete, if for any state \( h \) of \( \Sigma \), and for any event generator \( G \), the input-output map \( \mathcal{M}_{\Sigma, h, G} \) exists.

With the notation above, we are ready to present the first one of the main theorems.

Theorem 1: Assume \( \Sigma \) is a full-dimensional and complete PAHS. If \( \Sigma \) is observable, then for each \( q \in Q \), \( \mathcal{O}_{\Sigma, q} = \{0\} \).

Theorem 1 can be viewed as a direct extension of the results of [8]. Below we present a more tight necessary condition for observability.

### Definition 15 (Linear PAHS): A PAHS of the form (1) is called linear, if for the all \( q \in Q \),
Definition 16 (Weak observability): The PAHS $\Sigma$ is called weakly observable, if the following conditions hold.

(i) For each two states $s_1, s_2 \in Q$, $s_1 = s_2$ if and only if
\[ \forall v \in \Gamma^* : \lambda(\delta(s_1, v)) = \lambda(\delta(s_2, v)), \]
\[ \forall v \in \Gamma^* : M_{s_1}(w, v) = M_{s_2}(w, v) \quad (3) \]

(ii) For each $q \in Q$, the zero vector is the only element of the subspace $O_{\Sigma, q}$, i.e. $O_{\Sigma, q} = \{0\}$.

Later on we will show that weak observability is equivalent to observability of the LHS $H_\Sigma$ associated with $\Sigma$.

Theorem 2: Assume $\Sigma$ is a full-dimensional, linear and complete PAHS. If $\Sigma$ is observable, then it is weakly observable.

Remark 3: It is possible to extend Theorem 1–2 to hold for PAHSs which are not complete. To this end, one has to define input-output maps for the case when several consecutive events may occur with no time lag between them. In this paper we restrict attention to complete PAHSs only in order to avoid complicated notation.

The intuition behind Theorem 2 is the following. Since $(\Sigma, G)$ is a feedback interconnection of $H_\Sigma$ with $G$, if $(\Sigma, G)$ is observable, then so is $H_\Sigma$. In turn, weak observability of $\Sigma$ is equivalent to observability of $H_\Sigma$.

Remark 4: Weak observability of $\Sigma$ can be checked numerically, since weak observability of $\Sigma$ is equivalent to observability of the LHS associated with $\Sigma$, and by [9], the latter can be checked numerically.

Theorem 3 (Observability reduction): Any (hence not necessarily linear) PAHS $\Sigma$ of the form (1) can be transformed to a weakly observable, linear, and full dimensional PAHS $\Sigma_{o}$, such that

(a) For any discrete-event generator $G (\Sigma_{o}, G)$ realizes the same input-output map as $(\Sigma, G)$, i.e. $y_{\Sigma_{o}, G} = y_{\Sigma, G}$.

(b) $\dim \Sigma_{o} \leq (|Q|, \sum_{q \in Q} (n_{q} + 1))$. If $\Sigma$ is a linear full-dimensional PAHS, then $\dim \Sigma_{o} \leq \dim \Sigma$.

Moreover, $\Sigma_{o}$ can effectively be computed from $\Sigma$. Note that the PAHS $\Sigma_{o}$ above need not be complete. The intuition behind the theorem is as follows. It is always possible to convert a PAHS to a linear full-dimensional PAHS realizing the same input-output map, see Section V. Hence, without loss of generality we can assume that $\Sigma$ is linear and full dimensional. It is possible to convert the LHS $H_{\Sigma}$ of $\Sigma$ to an observable LHS $H_{\Sigma_{o}}$ which realizes the same input-output behavior. If in the feedback loop we replace $H_{\Sigma}$ with $H_{\Sigma_{o}}$, then we obtain a PAHS $\Sigma_{o}$ (see Fig. 2), which has the same input-output behavior as $\Sigma$, but the associated LHS of which is observable, i.e. $\Sigma_{o}$ which is weakly observable.

V. PROOF OF THE MAIN RESULT

In this section we present the proof of the main result. In §V-A we recall from [10], [9] the notion of LHSs. In §V-B we present the relationship between PAHSs and LHSs. In §V-C we present the behavior preserving transformation of a PAHS to a linear full dimensional one. Finally, in §V-D we sketch the proofs of the main results.

A. Linear hybrid systems

The current section is a review of [9], [10].

Definition 17 (Linear hybrid systems,[9], [10]): A linear hybrid system (abbreviated as LHS ) is a hybrid system without guards of the form
\[
\dot{x}(t) = A_{q(t)}x(t) + B_{q(t)}u(t), \quad y(t) = C_{q(t)}x(t)
\]
\[ q(t+) = \delta(q(t), \gamma(t)), \quad o(t) = \lambda(q(t)) \]
\[ x(t+) = M_{q(t+) \gamma(t), q(t)} x(t) \]
\[ h_{0} = (q_{0}, x_{0}) \quad (4) \]

• $q(t) \in Q$ is the discrete state at time $t$, and $Q$ is the finite set of discrete states (modes),
• $o(t) \in O$ is the discrete output at time $t$, and $O$ is the finite set of discrete outputs,
• $\gamma(t) \in \Gamma$ is discrete event at time $t$, and $\Gamma$ is the finite set of discrete events,
• $\delta : Q \times \Gamma \rightarrow Q$ is the discrete state-transition map,
• $\lambda : Q \rightarrow O$ is the discrete readout map,
• $A_{q} \in \mathbb{R}^{n_{q} \times n_{q}}$, $B_{q} \in \mathbb{R}^{n_{q} \times m}$, $C_{q} \in \mathbb{R}^{p \times n_{q}}$ are the matrices, and $X_{q} = \mathbb{R}^{n_{q}}$, $n_{q} > 0$, is the continuous state-space, of the linear system in $q \in Q$,
• $x(t) \in \mathbb{R}^{n_{q}(t)}$ is the continuous state at time $t$,
• $y(t) \in \mathbb{R}^{p}$, for $p > 0$, is the continuous output at time $t$, and $\mathbb{R}^{p}$ is the space of continuous outputs,
• $u(t) \in \mathbb{R}^{m}$, $m > 0$, is the continuous input at time $t$, and $\mathbb{R}^{m}$ is the space of continuous inputs,
• the matrices $M_{q(\gamma), q} \in \mathbb{R}^{n_{q}(q) \times n_{q}}$, $q \in Q$, $\gamma \in \Gamma$, specify the linear reset maps,
• $h_{0} = (q_{0}, x_{0})$ – initial state.

The state space $\mathcal{H}_{H}$ of $H$ is $\mathcal{H}_{H} = \bigcup_{q \in Q}(q) \times X_{q}$.

In the rest of this section, $H$ denotes a LHS of the form (4). The state and output of an LHS evolves as follows. If no discrete event occurs, the evolution is governed by the linear system of the current discrete state. As soon as a discrete event arrives, a discrete-state transition occurs, the continuous state is reset according to the reset map, and the system resumes its evolution according to the linear system of the new discrete state. Note that for LHSs discrete events are external inputs, and there are no guards. For the formal description, we need the following notion.
Definition 18 (Timed sequences): A timed sequence of events is a sequence
\[ w = (\gamma_1, t_1)(\gamma_2, t_2) \cdots (\gamma_k, t_k) \]
where \( \gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma, k \geq 0, \) and \( t_1, t_2, \ldots, t_k \in T. \) We denote the set of all such sequences by \( (\Gamma \times T)^*. \) If \( k = 0, \) then \( w \) is the empty sequence, and it is denoted by \( \epsilon. \)

The interpretation of \( w \) above is the following. The event \( \gamma_i \) took place after \( \gamma_{i-1} \) and \( t_i \) is the elapsed time between the arrival of \( \gamma_{i-1} \) and the arrival of \( \gamma_i. \) If \( i = 1, \) then \( t_1 \) is the arrival time of the first event \( \gamma_1. \)

Notation 6: Denote the set of inputs of an LHS by \( U = PC(T, R^m) \times (\Gamma \times T)^* \times T. \)

Definition 19 (State evolution): Consider a triple \( u = (u, w, t_{k+1}) \in U, \) where \( w \) is of the form (5). For a state \( h = (q, x) \in H_H \) of \( H, \) define the state \( \xi_H(h, u, w, t_{k+1}) \) reached from \( h \) with inputs \( (u, w, t_{k+1}) \) at time \( \sum_{j=1}^{k+1} t_j \) recursively on \( k \) as follows.

For \( k = 0, \) let \( q \) and \( x(t) \in \mathcal{X}_q \) be the solution of (6)
\[ \dot{x}(t) = A_q x(t) + B_q u(t) \]
with \( x(0) = x, \) and set \( \xi_H((h, u, \epsilon, t_1)) = (q, x(t_1)). \)

If \( v = (\gamma_1, t_1)(\gamma_2, t_2) \cdots (\gamma_{k-1}, t_{k-1}) \in (\Gamma \times T)^*, \) \( k > 0, \) the state \( \xi_H(h, v, u, t_k) = (q_k, x_k) \) is already defined, then set \( q_k = \delta(q_{k-1}, \gamma_k), \) and let \( z(t) \in \mathcal{X}_{q_k} \) be the solution of (7) with the initial condition \( z(0) = M_{q_k,q_{k-1}} x_k. \)
\[ \dot{z}(t) = A_{q_k} z(t) + B_{q_k} u(t + \sum_{i=1}^{k} t_j) \]
Set then \( \xi_H(h, v, u, t_{k+1}) = (q_k, z(t_{k+1})). \)

Note that in the case \( \xi_H(h, u, w, t_{k+1}) \) denotes the time which has passed since the arrival of the last event \( \gamma_k. \) Next, we will define the input-output behavior of LHSs induced by a state.

Definition 20 (Input-output maps): The input-output map \( \nu_{H,h} \) and the continuous input-output map \( \gamma_{H,h} \) of \( H \) induced by \( h \in H_H \) are maps
\[ \nu_{H,h} : U \rightarrow O \times R^p \] and \( \gamma_{H,h} : U \rightarrow R^\hat{p} \)
that for each \( u \in U, \) if \( (q, x) = \xi_H(h, u), \) then
\[ \nu_{H,h}(u) = (\lambda(q), \dot{C}_q x) \] and \( \gamma_{H,h}(u) = \dot{C}_q x. \)

Definition 21 (Realization): The LHS \( H \) is a realization of the map \( f : U \rightarrow O \times R^p \) if \( f \) equals the input-output map of \( H \) induced by the initial state, i.e. \( f = \nu_{H,h_0}. \)

Definition 22 (Observability): Two distinct states \( h_1 \neq h_2 \in H_H \) of the LHS \( H \) are indistinguishable, if the input-output maps induced by \( h_1 \) and \( h_2 \) are equal, i.e. \( \nu_{H,h_1} = \nu_{H,h_2}. \) The system is called observable, if it has no pair of distinct indistinguishable states.

B. Relationship between PAHSs and LHSs

We start by defining the LHS associated with a PAHS.

Definition 23 (LHS associated with PAHS): Consider a linear PAHS \( \Sigma \) of the form (1). Define the LHS \( H_{\Sigma} \) associated with \( \Sigma \) as the LHS of the form (4) such that the following holds.

- The set of discrete states, outputs, events, the discrete state-transition map and the discrete readout map of \( H_{\Sigma} \) are all the same as those of \( \Sigma. \)
- The matrices \( A_\Sigma, B_\Sigma \) and \( M_{\Delta(q, \gamma), \gamma, q}, q \in Q, \gamma \in \Gamma, \) of \( H_{\Sigma} \) are the same as those of \( \Sigma. \)
- \( \hat{p} = p|\Gamma|, \) and the readout matrix \( \dot{C}_\Sigma \) of \( H_{\Sigma} \) is the augmented readout map of \( \Sigma, \) as defined in (2).
- The initial state of \( H_{\Sigma} \) is the same as that of \( \Sigma. \)

Conversely, we can associate a PAHS with each LHS.

Definition 24 (PAHS associated with LHS): Consider an LHS \( H \) of the form (4), such that \( \hat{p} = p + |\Gamma|. \) The PAHS \( \Sigma_H \) associated with \( H \) is a PAHS of the form (1), such that the following holds.

- The set of discrete states \( Q, \) discrete outputs \( O, \) events \( \Gamma, \) the state-transition map \( \delta, \) and the readout map \( \lambda \) is the same for \( \Sigma_H \) as for \( H. \)
- For all \( q \in Q, \) the matrices \( A_q, B_q, M_{\Delta(q, \gamma), \gamma, q} \) of \( \Sigma_H \) are the same as those of \( H. \) The vectors \( a_q, c_q \) and \( m_{\Delta(q, \gamma), \gamma, q} \) are all zero.
- For all \( q \in Q, \) the matrix \( C_q \) of \( \Sigma_H \) is formed by the first \( p \) rows of \( C_q, i.e. \) \( C_q = \left[ C_q^T \ n_{1,q} \cdots n_{E,q} \right]^T. \)
- For each \( q \in Q, \) the polyhedron \( P_q \) of \( \Sigma_H \) is
\[ P_q = \bigcap_{\gamma \in \Gamma} \{ x \in R^p \mid n_{\gamma,q}^T x \leq 0 \} \]
where \( n_{\gamma,q}^T \) is the \( p + \gamma \)th row of \( \dot{C}_q. \)
- The initial state of \( \Sigma_H \) is the same as that of \( H. \)

Intuitively, \( \Sigma_H \) is obtained from \( H \) by defining the polyhedron for each discrete state \( q \in Q \) as the polyhedron, normal vectors of the exit facets of which correspond to the last \( |\Gamma| \) rows of the readout matrix \( \dot{C}_q \) of \( H. \) Notice that the correspondence of Definition 24 is dual to the one of Definition 23.

Lemma 1: With the notation of Definition 23, the LHS \( H_{\Sigma_H} \) associated with \( \Sigma_H \) equals \( H. \)

Next, we present a result which relates the state and output of PAHS with the state and output of the associated LHS. To this end, we need the following. Consider a DPAHS \( \hat{H}, \Sigma \) such \( \Sigma \) is linear and it is of the form (1). For an input \( u \in PC(T, R^m), \) state \( h \) of \( \Sigma, \) such that \( x_{\Sigma,H}, h \) exists, consider the domain \( T_{u,h} \) of \( x_{\Sigma,H}, h \). For any \( t \in [0, T_{u,h}], \) define the pair \( E V_{\Sigma,H}(h, u, t) = (s, \dot{s}) \in (\Gamma \times T)^* \times T \) such that \( s \) is the timed event sequence generated by \( \Sigma, \) on the interval \([0, t], \) if started in state \( h \) and fed input \( u, \) and no event occurs on \((t, l), \) formally. If \( n^* = 0, \) then \( \dot{l} = t \) and \( s = e; \) if \( n^* > 0, \) then \( s = (\gamma_1, t_1) \cdots (\gamma_k, t_k - t_{k-1}) \) and \( \dot{l} = t - t_{k-1}. \) Note that for \( t \in [0, T_{u,h}], \) \( E V_{\Sigma,H}(h, u, t) \) depends only on \( u_{H,h} \), and if \( u_{H,h}(u, EV_{\Sigma,H}(h, u, t)) = (o, (y, z)) \) with \( o \in O, y \in R^p, z \in R^E, \) then \( y_{\Sigma,H}(u)(t) = (o, y). \) Combining these remarks, we get the following.

Lemma 2: Assume that \( \Sigma_i, i = 1, 2 \) are linear PAHSs and let \( H_{\Sigma_i} \) be the LHSs associated with \( \Sigma_i, i = 1, 2. \) Assume that \( h_i \) is a state of \( \Sigma_i \) and that for any event generator \( G, y_{\Sigma_i,G,h_i}, \) exists, for i = 1, 2. If \( u_{H_{\Sigma_i}, h_i} = u_{H_{\Sigma_2}, h_2}, \) then for any event generator \( G, y_{\Sigma_1,G,h_1} = y_{\Sigma_2,G,h_2}. \)
Next, we state a result, which is interesting on its own right.

**Theorem 4:** Assume that $\Sigma$ is a linear, full-dimensional and complete PAHS and let $G$ be any event generator. If $\Sigma$ is observable, then the associated LHS $H_2$ is observable.

For the proof of Theorem 4, we need the following.

**Lemma 3:** Assume that $H$ is an LHS of the form (4). For any state $(q_i, x_i)$ of $H$, $i = 1, 2$, if $v_{H_1}(q_i, x_i) = v_{H_2}(q_i, x_2)$, then $v_{H_1}(q_i, 0) = v_{H_2}(q_i, 0)$. In addition, $v_{H_1}(q_i, x_i) = v_{H_2}(q_i, x_2)$ is equivalent to $y_{H_1}(q_i, x_1 - x_2)(0, w, t) = 0$ for all $w \in (T \times T)^t$, $t \in T$. Moreover, $y_{H_1}(q_i)(0, w, t)$ is linear in $x$.

The proof of Lemma 3 follows from the proof of Theorem 2, [10]. In addition, we need the following algorithmic result.

**Lemma 4:** If $P$ is a full dimensional polyhedron on $\mathbb{R}^n$ and $W$ is a proper (non-zero) linear subspace of $\mathbb{R}^n$, then there exist $x_1, x_2 \in P$ such that $x_1 - x_2 \in W$.

**Proof:** [Sketch of proof of Theorem 4] Assume that $\Sigma$ is observable, $H = H_2$, is not observable. The latter means that there exists a state $t_1 = (q_1, x_1), i = 1, 2$, such that $v_{H_1}(q_1, x_1) = v_{H_2}(q_1, x_2), x_2$ by Lemma 3, it then implies that there exists $v_{H_2}(q_1, 0)$ by linearity of $\Sigma$, $0 \in P_q$, for $i = 1, 2$. But then from Lemma 2 we obtain that $y_{\Sigma_1}(q_1, 0) = y_{\Sigma_2}(q_1, 0)$. Hence, if $q_1 \neq q_2$, we obtain that $\Sigma$ is not observable, which is a contradiction.

Assume that $q_1 = q_2$. Then by Lemma 3, $v_{H_1}(q_1, x_1) = v_{H_2}(q_2, x_1)$ is equivalent to $y_{H_1}(q_1, x_1 - x_2) = 0$. Denote by $W_q$ the set of all elements $x \in \mathbb{R}^n$ such that $y_{H_2}(q_2, x) = 0$. From Lemma 3 it follows that $W_q$ is a linear space and $x_1 - x_2 \in W_q, x_1 \neq x_2$ implies that $W_q$ is not trivial. Then by Lemma 4 we get that there exist $x_1, x_2 \in P_q, x_1 \neq x_2$ such that $x_1 - x_2 \in W_q$. But it implies that $y_{H_1}(q_1, x_1 - x_2) = 0$ and hence $y_{H_2}(q_2, x_1 - x_2) = y_{H_2}(q_2, x_2) = 0$. But the latter, together with the fact that $(q_1, x_1)$ and $(q_2, x_2)$ are both states of $\Sigma$ and Lemma 2 implies that $y_{\Sigma}(q_1, x_1) = y_{\Sigma_2}(q_1, x_1)$. But this contradicts to observability of $\Sigma$.  

**C. Conversion of a PAHS to a linear full-dimensional one**

In this section, $\Sigma$ is a PAHS of the form (1). First, we present the transformation of $\Sigma$ to a full dimensional PAHS.

**Definition 25:** Define the full-dimensional PAHS $F(\Sigma)$ associated with $\Sigma$ as follows.

- The discrete state and output sets, the discrete state-transition and readout maps of $\Sigma$ and $F(\Sigma)$ are identical.

- For each $q \in Q$, let $d_q$ be the dimension of the affine span of elements of $P_q$. Then the continuous state space of $F(\Sigma)$ in $q$ is $P_q$ and the affine system is

  $\dot{x}(t) = \hat{A}_q x(t) + \hat{B}_q u(t) + \hat{a}_q$

  $y(t) = \hat{C}_q x(t) + \hat{c}_q$

  The reset map $\hat{R}_q^{+,-,q}$ of $F(\Sigma)$ associated with $q \in Q, \gamma \in \Gamma, q^+ = \delta(q, \gamma)$ is defined as

  $\hat{R}_q^{+,-,q}(x) = M_q^{+,-,q} x + m_q^{+,-,q}$

  The matrices $\hat{A}_q, \hat{B}_q, \hat{C}_q, \hat{M}_q^{+,-,q}$, the polyhedron $\hat{P}_q$, and vectors $\hat{a}_q, \hat{c}_q, \hat{m}_q^{+,-,q}$ are defined as follows. Let $v_0, v_1, \ldots, v_d \in P_q$ be an affine basis of $P_q$. Then there exists $v_{d+1}, \ldots, v_n \notin P_q$ such that $v_i = v_i - v_0, i = 1, \ldots, n_q$, forms a basis of $\mathbb{R}^{n_q}$. Denote by $W_q$ the linear span $v_1, \ldots, d_q$ and let $S_q$ be the isomorphism $S_q : W_q \rightarrow \mathbb{R}^{d_q}$. Then $\mathbb{R}^{d_q}$ is the direct sum $W_q \oplus W_c$. Define the linear map $\Pi_q : \mathbb{R}^{n_q} \rightarrow \mathbb{R}^{d_q}$ by $\Pi_q(x) = S_q(x_1), x_x = x_1 + x_2, x_1 \in W_q$ and $x_2 \in W_c$. We identify $\Pi_q$ with the corresponding matrix and denote by $\Pi_q^{-1}$ the right inverse of $\Pi_q$. Then

  $\hat{A}_q = \Pi_q A_q \Pi_q^{-1}, \hat{a}_q = \Pi_q a_q, \hat{B}_q = \Pi_q B_q,$

  $\hat{C}_q = \Pi_q^{-1} c_q, \hat{m}_q^{+,-,q} = \Pi_q + M_q^{+,-,q} \Pi_q^{-1},$

  $\hat{P}_q = \Pi_q \{ x \mid n_q^{+,-,q} \Pi_q^{-1} x \leq b_q^{-}\gamma \}$

- The initial state $\hat{h}_0$ of $F(\Sigma)$ is $\hat{h}_0 = (q_0, \Pi_q(x_0))$. The above transformation preserves input-output behavior.

**Theorem 5:** With the notation of Definition 25, for any state $h = (q, x)$ of $\Sigma$, the state $\hat{h} = (q, \Pi_q(x))$ is a state of $F(\Sigma)$ and for any event generator $G$, $y_{\Sigma,G,h}$ exists if and only if $y_{F(\Sigma),G,h} exists$, and $y_{\Sigma,G,h} = y_{F(\Sigma),G,h}$. Next, we present the transformation of $\Sigma$ to a linear PAHS.

**Definition 26 (PAHS to linear PAHS 1):** Define the linear PAHS $L(\Sigma)$ associated with $\Sigma$ as follows.

- The set of discrete states and outputs, the discrete state-transition map and the discrete readout map $L(\Sigma)$ are the same as the corresponding items of $\Sigma$.

- For each $q \in Q$, the continuous state-space of $L(\Sigma)$ is $\mathbb{R}^{n_q,+}$ and the affine system is

  $\dot{x}(t) = \hat{A}_q x(t) + \hat{B}_q u(t)$

  $y(t) = \hat{C}_q x(t)$

  For each $\gamma \in \Gamma$, the reset map $\hat{R}_q^{+,-,q}$, $q^+ = \delta(q, \gamma)$, of $L(\Sigma)$ is linear, i.e. $R_q^{+,-,q}(x) = M_q^{+,-,q} x$. Here,

  $\hat{A}_q = \begin{bmatrix} A_q & a_q \\ 0 & 0 \end{bmatrix}, \hat{B}_q = \begin{bmatrix} B_q \\ 0 \end{bmatrix}$

  $\hat{C}_q = [C_q \ c_q]$

  $\hat{M}_q^{+,-,q} = \begin{bmatrix} M_q^{+,-,q} & m_q^{+,-,q} \\ 0 & 1 \end{bmatrix}$

  $\hat{P}_q = \bigcap_{\gamma \in \Gamma} \{ x \mid \hat{n}_q^{+,-,q} \leq b_q^{-}\gamma \}$

- The initial state of $L(\Sigma)$ is $\hat{h}_0 = (q_0, (x_0, 1))$. The intuition behind the construction of $L(\Sigma)$ is as follows. In order to encode the affine component of the continuous dynamics of $\Sigma$, for all discrete states an additional continuous state component is added. Hence, the state $(q, x, 1)$ of $L(\Sigma)$ corresponds to the state $(q, x)$ of $\Sigma$.

**Theorem 6:** Using the notation of Definition 26, $L(\Sigma)$ is a linear PAHS. If $\Sigma$ is full dimensional, then so is $L(\Sigma)$. For any state $h = (q, x)$ of $\Sigma$, $\hat{h} = (q, x, 1)$ is a state of $L(\Sigma)$, and for any event generator $G$, $y_{L(\Sigma),G,h}$ exists if and only if $y_{\Sigma,G,h}$ exists, and $y_{L(\Sigma),G,h} = y_{\Sigma,G,h}$.

**D. Proof of Theorem 1 – 3**

The proofs of Theorem 1 – 3 relies on the following result.

**Theorem 7:** A linear PAHS $\Sigma$ is weakly observable if and only if the associated LHS $H_2$ is observable.
The proof of Theorem 7 can directly be obtained by substituting the definition of $M_q(w, v)$ and $O_{\Sigma, q}$ into Theorem 2 of [10].

Proof: [Proof of Theorem 2] Consider the LHS $H_2$ associated with $\Sigma$. From Theorem 4 it follows that if $\Sigma$ is observable, then $H_2$ is observable, and hence by Theorem 7 $\Sigma$ is weakly observable.

Proof: [Proof of Theorem 1] Consider the linear PAHS $L(\Sigma)$ associated with $\Sigma$ and let $H = H_1(\Sigma)$ be the LHS associated with $L(\Sigma)$. Notice that if $\Sigma$ is full dimensional, then so is $L(\Sigma)$. Assume that for some $q \in Q$, there exists $O_{\Sigma, q} \neq 0$. Since $P_q$ is a full dimensional, from Lemma 4 we get that there exists $x_1, x_2 \in P_q$ such that $x_1 - x_2 \in O_{\Sigma, q}$.

Denote by $W$ the subset of $\mathbb{R}^{n_q+1}$ formed by vectors of the form $(x^T, 0)^T$, $x \in \mathbb{R}^{n_q}$. Denote $O_{H, q} = O_{(\Sigma), q}$. Note that $O_{H, q}$ is the observability subspace of $H$, as defined in [10], [9]. Note that $O_{H, q} \cap W = \{(z^T, 0)^T \mid z \in O_{\Sigma, q}\}$. Hence, then $0 \neq ((x_1 - x_2)^T, 0) \in O_{H, q} \cap W$. If we set $x_i = (x_i^T, 1)^T$, $i = 1, 2$, then $x_1 - x_2 = ((x_1 - x_2)^T, 0) \in O_{H, q}$. Recall from [10] that $\hat{x}_1 - \hat{x}_2 \in O_{H, q}$ in fact implies that $y_{H, q}(\hat{x}_1 - \hat{x}_2) = 0$. By Lemma 3 the latter implies that $v_{H, h_1} = v_{H, h_2}$. $\hat{h}_1 = (q, \hat{x}_i), i = 1, 2$. Then by Lemma 2, $Y_{L(\Sigma), G, h_1} = Y_{L(\Sigma), G, h_2}$, for any event generator $G$. Since by Theorem 6, $\hat{y}_{L(\Sigma), G, \hat{h}_1} = \hat{y}_{L(\Sigma), G, h_2}, i = 1, 2$, where $h_1 = (q, x_i), i = 1, 2$, we get that $\hat{y}_{L(\Sigma), G, \hat{h}_1} = \hat{y}_{L(\Sigma), G, h_2}$ and $\hat{h}_1 \neq \hat{h}_2$. But this contradicts the observability of $\Sigma$.

Proof: [Sketch of the proof of Theorem 3] Consider a PAHS $\Sigma$. Transform it to a full dimensional PAHS $\Sigma_1 = F(\Sigma)$. Transform $\Sigma_1$ to a linear (and full dimensional) PAHS $\Sigma_2 = L(\Sigma_1)$. By Theorem 6 and Theorem 5, if $h_0$ is the initial state of $\Sigma$ and $h_0^\Sigma$ is the initial state of $\Sigma_2$, then for any event generator $G$, $\hat{y}_{L(\Sigma), G, h_0} = \hat{y}_{L(\Sigma), G, h_0^\Sigma}$.

Consider the LHS $H$ associated with $\Sigma_1 = \Sigma_0$, $i.e., \Sigma_0$ is the PAHS associated with $H_o$. Since $H_o$ is observable, we get that $\Sigma_0$ is observable. In addition, $\dim \Sigma_0 = \dim H_o = \dim \Sigma_2$ and $\dim \Sigma_2 \leq \dim (p, r + p)$ where $(p, r) = \dim \Sigma_1$ and $\dim \Sigma_1 \leq \dim \Sigma$. Moreover, if $\Sigma$ is a linear PAHS, then so is $\Sigma_0$ and $\Sigma_2$, and then instead of $\Sigma_2$ we can take simply $\Sigma_1$, $i.e., \dim \Sigma_2 = \dim \Sigma_1 \leq \Sigma_0$. It is also easy to see that $\Sigma_0$ is full dimensional. To this end, recall from [10], [9] the definition of an LHS morphism and recall that there exists a surjective LHS morphism $S = (S_D, S_C) : H \rightarrow H_o$. The existence and surjectivity of $S$ implies that $S_D$ is a map $S_D : Q \rightarrow Q_o$, and $S_C$ is a linear map $S_C : \bigoplus_{q \in Q} X_q \rightarrow \bigoplus_{q \in Q} X_{q_0}$, where $Q_o$ is the set of discrete states of $H_o$ and $X_{q_0}$ is the continuous state-spaces of $H_o$ associated with discrete state $q_0 \in Q_o$. In addition, it holds that $S_C(X_q) \subseteq X_{q_0}$ and $\hat{C}_q S_C = \hat{C}_{S_D(q)} q \in Q$, where $\hat{C}_{S_D(q)}$ is the output matrix of $H_o$ associated with the discrete state $S_D(q)$. Hence, for all $q \in S_D^{-1}(q_0)$ and for any $c \in \Gamma$, if $\hat{n}_{q_0} q, \gamma$ denotes the $\gamma$th row of $\hat{C}_{q_0}$, then $\hat{n}_{q_0} S_C q, \gamma = n_{q_0} q, \gamma$, since $n_{q_0} q, \gamma$ is the $\gamma$th row of $\hat{C}_q$.

The proof of Theorem 7 can directly be obtained by substituting the definition of $M_q(w, v)$ and $O_{\Sigma, q}$ into Theorem 2 of [10].

Proof: [Proof of Theorem 2] Consider the LHS $H_2$ associated with $\Sigma$. From Theorem 4 it follows that if $\Sigma$ is observable, then $H_2$ is observable, and hence by Theorem 7 $\Sigma$ is weakly observable.

VI. CONCLUSIONS

We presented necessary conditions for observability of piecewise-affine hybrid systems, and observability reduction algorithm. The latter transforms a piecewise-affine system to a one which satisfies the necessary conditions. Future research is directed towards obtaining necessary and sufficient conditions for observability of piecewise-affine hybrid systems and an observability reduction algorithm.

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