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On the partial realization problem

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***Abstract*— We consider here a two sided interpolation problem where we want to minimize the degree of the interpolant. We show that this degree is given by the rank of a particular solution to a Sylvester equation which, in some particular cases becomes a Löwner or a Hankel matrix. We consider an application to the usual partial realization problem. The results are quite general and no particular assumption on the location of the interpolating nodes are needed.**

I. Introduction and Preliminaries

We consider here a two sided interpolation problem where we allow non disjoint interpolation nodes. We first consider the case when the interpolation points are disjoint from the poles of the interpolating function and show then how this restriction can be lifted. The problem has a long history (starting, in some sense, with the Ho-Kalman algorithm, see [8]) and was investigated by Rissanen [10] , Gragg and Lindquist [7] and others. In the interpolation formulation it was studied by Anderson and Antoulas [1] using Löwner matrices and later by Anotoulas, Ball, Kang and Willems [2] using linear fractional transformations. An approach which led to these results was developed in a special case by Kimura [9] and Georgiou [5] and generalized by the authors We show how a state space approach to the problem yields simple formulas for constructing the interpolants which do not require a specific structure of the interpolation nodes (e.g. all equal or all disjoint) and allows for a generalization to the case when the interpolant has poles also at the interpolation nodes.

If M is a complex matrix, Tr shall denote its trace, M^T its transpose and M^* its transpose conjugate. $\sigma(M)$ denotes its spectrum. The inclusion $\sigma(M_1) \subset^* \sigma(M_2)$ expresses the fact the spectrum of M_1 forms a subset of that of M_2 including multiplicities.

Let F be a rational $p \times m$ matrix of McMillan degree N with realization $F(z) = D + C\,(sI - A)^{-1}\,B$. We are going to use Rosenbrock's notation

$$F \sim \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) .$$

We consider here the problem of constructing a a $p \times m$ interpolating function Q given left and a right set of interpolants. The left set will be determined by the matrices $W_1, V_1, \mathcal{A}_1$ of dimension $n_1 \times m, n_1 \times p$ and $n_1 \times n_1$, respectively and, if the spectra of $\mathcal{A}_1$ and the poles of Q are disjoint, the interpolation conditions can be expressed as

$$(sI - \mathcal{A}_1)^{-1}[W_1 Q(s) - V_1]$$

being analytic in $\sigma(\mathcal{A}_1)$. Similarly, let $W_2, V_2, \mathcal{A}_2$ be matrices of dimension $p \times n_2, m \times n_2$ and $n_2 \times n_2$, respectively. The right interpolation condition will write as

$$[Q(s)W_2 - V_2](sI - \mathcal{A}_2)^{-1} \tag{1}$$

being analytic in $\sigma(\mathcal{A}_2)$. If the spectra of $\mathcal{A}_1$ and $\mathcal{A}_2$ are not disjoint, it is well known (see e.g. [3]) that an extra joint condition is needed (and it will be given in terms of an $n_1 \times n_2$ matrix H below).

It should be noted that, for example, if Q is scalar and $\mathcal{A}_2$ is the diagonal $\mathrm{diag}\{s_1, ...s_n\}$, then (1) can be written in the familiar form

$$Q(s_i)w_i = v_i \qquad i = 1, ...n$$

where w_i, v_i are the entries of W_2 and V_2, respectively.

It is well-known (see e.g. [3]) that all solutions of this problem can be given using a rational fractional representation defined by a J-inner function. In [2] these techniques are used to obtain a characterization of all minimal solutions. Here we use state space techniques which greatly simplify the interpolant construction. Moreover, we can also have interpolation nodes at the poles of Q. The paper is structured as follows: in Section II we consider the general interpolation problem. First, in Subsection II-A we characterize the interpolant when the spectra of $\mathcal{A}_1, \mathcal{A}_2$ do not intersect the poles of Q. The case when these assumptions are no longer true is treated in Subsection II-B. In Section III we apply the results to the partial realization problem. In Section IV we connect with the Ho-Kalman algorithm.

II. Main results

A. The case of disjoint spectra

In case the spectra of $\mathcal{A}_1, \mathcal{A}_2$ do not intersect the poles of Q the interpolation construction is relatively easy.

Theorem 2.1: Let Q be an $m \times p$ proper rational function with minimal realization $Q(s) = D + C(sI - A)^{-1}B$, $\mathcal{A}_1, \mathcal{A}_2$ be square matrices of dimensions n_1, n_2 respectively such that their spectra do not intersect with the poles of Q, W_1, W_2 be of dimensions $n_1 \times m$ and $p \times n_2$.

Then, given an $n_1 \times n_2$ matrix H, the rational function Q satisfies the interpolation problem

$$\frac{1}{2\pi i}\int_\Gamma (sI - \mathcal{A}_1)^{-1} W_1 Q(s) W_2 (sI - \mathcal{A}_2)^{-1} ds = H \tag{2}$$

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for any closed curve Γ containing $\sigma(\mathcal{A}_1) \cup \sigma(\mathcal{A}_2)$ but not the poles of Q, if and only if the solutions Y_1, Y_2 to

$$-\mathcal{A}_1 Y_1 + Y_1 A + W_1 C = 0 \quad (3)$$

$$AY_2 - Y_2\mathcal{A}_2 + BW_2 = 0 \quad (4)$$

satisfy the condition

$$Y_1 Y_2 = -H \; . \quad (5)$$

In this case, defining the matrices V_1 and V_2 of size $n_1 \times p$ and $m \times n_2$ as follows

$$V_1 := Y_1 B + W_1 D \quad (6)$$

$$V_2 := CY_2 + DW_2 \quad (7)$$

we have that

$$(sI - \mathcal{A}_1)^{-1} (W_1 Q(s) - V_1) = -Y_1 (sI - A)^{-1} B \quad (8)$$

and

$$(Q(s)W_2 - V_2)(sI - \mathcal{A}_2)^{-1} = -C(sI - A)^{-1} Y_2 \quad (9)$$

Proof. Suppose Q satisfies (2). Notice first that, in view of the assumptions on the spectra of $\mathcal{A}_1, \mathcal{A}_2$ and A, the solutions Y_1, Y_2 to (3) and (4) exist and are unique.

Thus, from (4) and (7), we obtain

$$\begin{aligned}
& [Q(s)W_2 - V_2](sI - \mathcal{A}_2)^{-1} \\
= \;& [(D + C(sI - A)^{-1}B)W_2 - V_2](sI - \mathcal{A}_2)^{-1} \quad (10)\\
= \;& [DW_2 - V_2](sI - \mathcal{A}_2)^{-1} \\
& + C(sI - A)^{-1} BW_2 (sI - \mathcal{A}_2)^{-1} \\
= \;& [DW_2 - V_2](sI - \mathcal{A}_2)^{-1} \\
& + C(sI - A)^{-1}(-AY_2 + Y_2\mathcal{A}_2)(sI - \mathcal{A}_2)^{-1} \\
= \;& [DW_2 - V_2](sI - \mathcal{A}_2)^{-1} \\
& - C(sI - A)^{-1} Y_2 + CY_2 (sI - \mathcal{A}_2)^{-1} \\
= \;& -C(sI - A)^{-1} Y_2 \quad (11)
\end{aligned}$$

which is (9). Formula (8) is proven similarly. Therefore, we can write

$$\begin{aligned}
& \frac{1}{2\pi i}\int_\Gamma (sI - \mathcal{A}_1)^{-1} W_1 Q(s) W_2 (sI - \mathcal{A}_2)^{-1} ds \\
&= \frac{1}{2\pi i}\int_\Gamma (sI - \mathcal{A}_1)^{-1} W_1 [Q(s)W_2 - V_2](sI - \mathcal{A}_2)^{-1} ds \\
&= -\frac{1}{2\pi i}\int_\Gamma (sI - \mathcal{A}_1)^{-1} W_1 C (sI - A)^{-1} Y_2 ds \\
&\qquad \text{in view of (9)} \\
&= -\frac{1}{2\pi i}\int_\Gamma (sI - \mathcal{A}_1)^{-1}[-Y_1 A + \mathcal{A} Y_1](sI - A)^{-1} Y_2 ds \\
&\qquad \text{in view of (3)} \\
&= -\frac{1}{2\pi i}\int_\Gamma [(sI - \mathcal{A}_1)^{-1} Y_1 Y_2 - Y_1 (sI - A)^{-1} Y_2] ds \\
&= -Y_1 Y_2 = H
\end{aligned}$$

Conversely, if (5) is satisfied, backtracking the above argument, it is easy to see that Q solves (2). ■

Under the same assumption, we can now get a lower bound on the degree of Q

Corollary 2.1: Suppose the spectra of A and $\mathcal{A}_1, \mathcal{A}_2$ do not intersect. Then the degree of Q is greater than or equal to the rank of H.

Proof. We show that the span of Y_2 is in the controllability subspace of (A, B): using the P-B-H test, we see that, if ξ is orthogonal to the controllability subspace of (A, B), it is $\xi^* B = 0$ and $\xi^* A = \alpha \xi^*$ where α is an eigenvalue of A. Thus, multiplying (4) by ξ^*, we obtain:

$$\xi^* A Y_2 = \alpha \xi^* Y_2 = \xi^* Y_2 \mathcal{A}_2$$

But this would imply that α is an eigenvalue of $\mathcal{A}_2$, which contradicts the assumption unless $\xi^* Y_2 = 0$, as claimed. Similarly, the kernel of Y_1 contains the non-observability subspace.

Now, the rank of H is equal to the dimension of the subspace containing those vectors in the range of Y_2 which are orthogonal to the kernel of Y_1. This subspace is obviously contained in those part of the controllability subspace which is orthogonal to the non-observability subspace the dimension of which gives the McMillan-degree of Q, concluding the proof of the corollary.

■

Remark. Let us point out that in the case when the spectra of $\mathcal{A}_1$ and $\mathcal{A}_2$ are disjoint from the set of poles of Q then obviously

$$\int_\Gamma Q(s) W_2 (sI - \mathcal{A}_2)^{-1} = V_2 \quad (12)$$

and

$$\int_\Gamma (sI - \mathcal{A}_2)^{-1} W_1 Q(s) = V_1 \quad (13)$$

and these interpolation equation determine uniquely Y_1, Y_2.

Corollary 2.2: Suppose Y_1 is left invertible and Y_2 is right invertible. Then Q has realization

$$Q = \left[\begin{array}{c|c} Y_1^{-L}\mathcal{A}_1 Y_1 - Y_1^{-L} W_1 C & Y_1^{-L}(V_1 - W_1 D) \\ \hline C & D \end{array}\right] \quad (14)$$

where $C = (V_2 - DW_2)Y_2^{-R}$ Similarly, Q has realization

$$Q = \left[\begin{array}{c|c} Y_2\mathcal{A}_2 Y_2^{-R} - BW_2 Y_2^{-R} & B \\ \hline (V_2 - DW_2)Y_2^{-R} & D \end{array}\right] \quad (15)$$

where $B = Y_1^{-L}(V_1 - W_1 D)$.

Proof. Realizations (14) and (15) follow immediately from (3), (7) and (4), (6), respectively. ■

Corollary 2.3: Let $\mathcal{A}_1$ and $\mathcal{A}_2$ have disjoint spectra. Then H in (5) is the unique solution to

$$\mathcal{A}_1 H - H\mathcal{A}_2 + V_1 W_2 - W_1 V_2 = 0$$

Proof. Notice that we can write (3), (4), (6) and (7) as

$$[Y_1, W_1]\begin{bmatrix} A & B \\ C & D \end{bmatrix} = [\mathcal{A}_1 Y_1, V_1]$$

and

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} Y_2 \\ W_2 \end{bmatrix} = \begin{bmatrix} Y_2\mathcal{A}_2 \\ V_2 \end{bmatrix}$$

Multiplying the first equation on the right by $\begin{bmatrix} Y_2 \\ W_2 \end{bmatrix}$ and the second on the left by $[Y_1, W_1]$, we readily obtain:

$$\mathcal{A}_1 Y_1 Y_2 + V_1 W_2 = Y_1 Y_2 \mathcal{A}_2 + W_1 V_2 \qquad (16)$$

which, in view of (5), yields (16). Since the spectra of $\mathcal{A}_1$ and $\mathcal{A}_2$ are disjoint, the solution is unique. ■

Notice that, if the spectra of $\mathcal{A}_1$ and $\mathcal{A}_2$ are simple, we can diagonalize these matrices and this results into $Y_1 Y_2$ being a Löwner matrix. Realizations (14) and (15) thus provide an alternative approach to interpolation to the one presented in [1].

Similarly, if $\mathcal{A}_2 = -\mathcal{A}_1^*$, $W_2 = -V_1^*$ and $W_1 = -V_2^*$, then H satisfies

$$\mathcal{A}_1 H + H\mathcal{A}_1^* + W_1 W_1^* - V_1 V_1^* = 0$$

which is the equation satisfied by the Pick matrix. If Y_1 and Y_2 are square, realization (14) becomes

$$Q = \left[\begin{array}{c|c} -\mathcal{A}_1^* - H^{-1}(V_1 - W_1 D) W_2 & H^{-1}(V_1 - W_1 D) \\ \hline V_2 - D W_2 & D \end{array}\right] \qquad (17)$$

If D is, for instance, tall, then $H^{-1}(V_1 - W_1 D) = -D(V_2 - D W_2)^*$ and thus (17) becomes the well known formula for an all-pass realization of a function interpolating $(\mathcal{A}_1, W_1, V_1)$ (see e.g. [4]).

B. Confluent spectra

If the assumption that the spectra of $\mathcal{A}_1, \mathcal{A}_2$ do not intersect the poles of Q is no longer valid, we can still characterize the interpolants using matrix conditions. The interpolation conditions, though have to be modified in order to accommodate the simultaneous presence of zeros and pole in the same node.

Theorem 2.2: Let Q be an $m \times p$ proper rational function with minimal realization $Q(s) = D + C\,(sI - A)^{-1}\,B$, and $\mathcal{A}_1, \mathcal{A}_2$ be square matrices of dimensions n_1, n_2, respectively, W_1, W_2 be of dimensions $n_1 \times m$ and $p \times n_2$. Let, moreover, H and V_2 be $n_1 \times n_2$ and $m \times n_2$.

Then, if there exist matrix polynomials β and δ such that Q satisfies the interpolation problem

$$\begin{aligned} &(Q(s)W_2 - V_2)\,(sI - \mathcal{A}_2)^{-1} + Q(s)\beta(s) \\ &\qquad \text{is analytic on } \sigma(\mathcal{A}_1) \cup \sigma(\mathcal{A}_2) \qquad (18) \\ &(sI - \mathcal{A}_1)^{-1}\left(W_1\,(Q(s)W_2 - V_2)\,(sI - \mathcal{A}_2)^{-1} + H\right) \\ &\quad + \delta(s)Q(s)\beta(s) \\ &\qquad \text{is analytic on } \sigma(\mathcal{A}_1) \cup \sigma(\mathcal{A}_2) \qquad (19) \end{aligned}$$

then there exist solutions Y_1, Y_2 to

$$-\mathcal{A}_1 Y_1 + Y_1 A + W_1 C = 0 \qquad (20)$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Y_2 \\ W_2 \end{bmatrix} = \begin{bmatrix} Y_2 \mathcal{A}_2 \\ V_2 \end{bmatrix} \qquad (21)$$

satisfying the condition

$$Y_1 Y_2 = -H. \qquad (22)$$

Conversely, if there exist solutions of (20), (21) and (22) then there exist matrix polynomials β, δ such that the functions in (18) and (19) are matrix polynomials.

Remark. If the spectra of $\mathcal{A}_1$ and $\mathcal{A}_2$ do not intersect with the poles of Q the interpolation conditions (18) and (19) imply that

$$\frac{1}{2\pi i}\int_\Gamma Q(s) W_2 (sI - \mathcal{A}_2)^{-1} ds = V_2 \qquad (23)$$

and

$$\frac{1}{2\pi i}\int_\Gamma (sI - \mathcal{A}_1)^{-1} W_1 Q(s) W_2 (sI - \mathcal{A}_2)^{-1} ds = H \qquad (24)$$

where Γ is a closed curve containing $\sigma(\mathcal{A}_1) \cup \sigma(\mathcal{A}_2)$ but not the poles of Q.

This is immediate from the observation that if T is constant matrix of dimension $n_1 \times n_2$ and Γ is any closed curve around the spectra of $\mathcal{A}_1, \mathcal{A}_2$, then

$$\int_\Gamma (sI - \mathcal{A}_1)^{-1} T (sI - \mathcal{A}_2)^{-1} ds = 0 \qquad (25)$$

In fact, this is just the $(2,1)$-block in the integral

$$\frac{1}{2\pi i}\int_\Gamma \left(sI - \begin{bmatrix} \mathcal{A}_2 & \\ T & \mathcal{A}_1 \end{bmatrix}\right)^{-1} ds$$

which is known to be the identity.

Furthermore, defining V_1 as

$$V_1 = Y_1 B + W_1 D \qquad (26)$$

we obtain that

$$\frac{1}{2\pi i}\int_\Gamma (sI - \mathcal{A}_2)^{-1} W_1 Q(s) ds = V_1\ . \qquad (27)$$

Proof. Suppose Q satisfies (18). Define

$$Y_2 = \int_\Gamma (sI - A)^{-1} B \left(W_2 (sI - \mathcal{A}_2)^{-1} + \beta(s)\right)\ ,$$

where now Γ is a closed curve containing $\sigma(\mathcal{A}_1) \cup \sigma(\mathcal{A}_2)$ but separating the remaining poles of Q.

Then (18) implies that

$$\begin{aligned} &C Y_2 + D W_2 \\ &= \int_\Gamma Q(s) W_2 (sI - \mathcal{A}_2)^{-1} + (Q(s) - D)\beta(s) ds = V_2\ . \end{aligned}$$

Furthermore

$$\begin{aligned} & A Y_2 - Y_2 \mathcal{A}_2 \\ = \;& -B W_2 + \int_\Gamma s(sI - A)^{-1} B \left(W_2 (sI - \mathcal{A}_2)^{-1} + \beta\right) \\ & + \int_\Gamma (sI - A)^{-1} B W_2 \\ & - \int_\Gamma (sI - A)^{-1} B W_2 (sI - \mathcal{A}_2)^{-1} s ds \\ & + \int_\Gamma (sI - A)^{-1} B \beta(s) \mathcal{A}_2 ds \\ = \;& -B W_2 + \int_\Gamma (sI - A)^{-1} B \left(W_2 + \beta(s)(sI - \mathcal{A}_2)\right) ds\ . \end{aligned}$$

Now

$$\begin{aligned} &CA^k(sI-A)^{-1}B\left(W_2+\beta(s)(sI-\mathcal{A}_2)\right)\\ &=C\left(A^k-s^kI\right)(sI-A)^{-1}B\left(W_2+\beta(s)(sI-\mathcal{A}_2)\right)\\ &\quad+s^k\left[\left(Q(s)W_2-V_2\right)(sI-\mathcal{A}_2)^{-1}+Q(s)\beta(s)\right]\\ &\qquad\cdot(sI-\mathcal{A}_2)\\ &\quad-s^kD(W_2+\beta(s)(sI-\mathcal{A}_2))+s^kV_2 \end{aligned}$$

giving that it is analytic inside Γ. The observability of (C,A) proves that the functions $(sI-A)^{-1}B\left(W_2+\beta(s)(sI-\mathcal{A}_2)\right)$ is analytic, as well. Consequently

$$AY_2-Y_2\mathcal{A}_2=-BW_2\ ,$$

proving (21).

Continuing with the converse statement of this part, if Y_2 is any solution of (21) then the controllability of (A,B) implies that there exist polynomials α and β' such that

$$Y_2=(sI-A)\alpha+B\beta'\ .$$

Now

$$\begin{aligned} &\left(Q(s)W_2-V_2\right)(sI-\mathcal{A}_2)^{-1}+Q(s)\beta'(s)\\ &\quad=\left(DW_2-V_2+C(sI-A)^{-1}(Y_2\mathcal{A}_2-AY_2)\right)\\ &\qquad\cdot(sI-\mathcal{A}_2)^{-1}+Q(s)\beta'(s)\\ &\quad=\left(-CY_2+C(sI-A)^{-1}\left((sI-A)Y_2\right.\right.\\ &\qquad\left.\left.-Y_2(sI-\mathcal{A}_2)\right)\right)(sI-\mathcal{A}_2)^{-1}+Q(s)\beta'(s)\\ &\quad=-C(sI-A)^{-1}Y_2+Q\beta'=-C\alpha+D\beta'\ . \end{aligned}$$

i.e. it is a matrix polynomial.

The previous consideration shows that $Q(\beta-\beta')$ is analytic on $\sigma(\mathcal{A}_1)\cup\sigma(\mathcal{A}_2)$ giving that in (19) instead of β we might write β', i. e.

$$\begin{aligned} &(sI-\mathcal{A}_1)^{-1}\left(W_1\left(Q(s)W_2-V_2\right)(sI-\mathcal{A}_2)^{-1}+H\right)+\delta Q\beta'\\ &\qquad\text{is analytic on } \sigma(\mathcal{A}_1)\cup\sigma(\mathcal{A}_2)\ . \end{aligned}$$

Here now

$$\delta Q\beta'=\delta D\beta'-\delta C\alpha+\delta C(sI-A)^{-1}Y_2\ ,$$

and

$$\begin{aligned} &\left(Q(s)W_2-V_2\right)(sI-\mathcal{A}_2)^{-1}\\ &=\left(C(sI-A)^{-1}(Y_2\mathcal{A}_2-AY_2)\right.\\ &\qquad\left.+DW_2-V_2\right)(sI-\mathcal{A}_2)^{-1}\\ &=-C(sI-A)^{-1}Y_2\ . \end{aligned}$$

Thus (19) can be formulated as

$$\begin{aligned} &(sI-\mathcal{A}_1)^{-1}\left(-W_1C(sI-A)^{-1}Y_2+H\right)\\ &\qquad+\delta(s)C(sI-A)^{-1}Y_2 \end{aligned}$$

is analytic on $\sigma(\mathcal{A}_1)\cup\sigma(\mathcal{A}_2)$.

Define now Y_1 as follows:

$$Y_1=\int_\Gamma\left(\delta(s)-(sI-\mathcal{A}_1)^{-1}W_1\right)C(sI-A)^{-1}\ .$$

Then obviously

$$Y_1Y_2=-H\ .$$

Furthermore, similar computation as above gives that

$$Y_1A-\mathcal{A}_1Y_1=W_1C+\int_\Gamma\left((sI-\mathcal{A}_1)\delta(s)-W_1\right)C(sI-A)^{-1}\ .$$

Now using the reachability of (A,B) it can be proved that the integrand in the second term is analytic on $\sigma(\mathcal{A}_1)\cup\sigma(\mathcal{A}_2)$.

Thus equations (22) and (20) hold.

To prove the converse statement for (19) if Y_1 is a solution of (20) then the observability of (C,A) implies that there exist matrix polynomials γ and δ' such that

$$Y_1=\gamma(sI-A)-\delta'C\ .$$

Then

$$\begin{aligned} &(sI-\mathcal{A}_1)^{-1}\left(W_1\left(Q(s)W_2-V_2\right)(sI-\mathcal{A}_2)^{-1}+H\right)\\ &\qquad+\delta'(s)Q(s)\beta'(s)\\ &\quad=(sI-\mathcal{A}_1)^{-1}\left(-W_1C(sI-A)^{-1}Y_2+H\right)\\ &\qquad+\delta'(s)Q(s)\beta'(s)\\ &\quad=(sI-\mathcal{A}_1)^{-1}\left(\left((sI-\mathcal{A}_1)Y_1\right.\right.\\ &\qquad\left.\left.-Y_1(sI-A)\right)(sI-A)^{-1}Y_2+H\right)+\delta'(s)Q(s)\beta'(s)\\ &\quad=Y_1(sI-A)^{-1}Y_2+\delta'Q\beta'\\ &\quad=(\gamma(sI-A)-\delta'C)(sI-A)^{-1}((sI-A)\alpha+B\beta')\\ &\qquad+\delta'Q\beta'\\ &\quad=\gamma(sI-A)\alpha-\delta'C\alpha+\gamma B\beta'+\delta'D\beta' \end{aligned}$$

which is a matrix polynomial, proving the converse statement. ■

Note that if $V_1=Y_1C+W_1D$ as above then it can be proved similarly as above that there exists a matrix polynomial δ' such that

$$(sI-\mathcal{A}_1)^{-1}\left(W_1Q(s)-V_1\right)+\delta'(s)Q(s)$$

is a matrix polynomial, as well, especially it is analytic on $\sigma(\mathcal{A}_1)\cup\sigma(\mathcal{A}_2)$.

III. Partial Realization

Suppose we now have a multivariate partial realization problem. That is, assuming $Q(s)$ is analytic around s_0, we want to find

$$Q(s)=\sum_{i=0}^{2n-1}R_i(s-s_0)^i+o(s-s_0)^{2n} \qquad (28)$$

where R_i are $m\times p$ real matrices we assume that Q has no poles in s_0. Notice that, if $Q(s)=D+C(sI-A)^{-1}B$, then developing in power series aorund s_0, we obtiain

$$\begin{aligned} &R_0=C(s_0I-A)^{-1}B+D\\ &R_n=(-1)^{n-1}C(s_0I-A)^{-n}B \qquad \text{for } n>0 \end{aligned} \qquad (29)$$

Then, setting

$$\mathcal{A}_1 := \begin{bmatrix} I_p s_0 & & & \\ I_p & I_p s_0 & & \\ & \ddots & \ddots & \\ & & I_p & I_p s_0 \end{bmatrix} \quad (30)$$

$$W_1 := \begin{bmatrix} I_p \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad V_1 := \begin{bmatrix} R_0 \\ R_1 \\ \vdots \\ R_{n-1} \end{bmatrix}$$

$$\mathcal{A}_2 := \begin{bmatrix} I_m s_0 & I_m & & \\ & I_m s_0 & \ddots & \\ & & \ddots & I_m \\ & & & I_m s_0 \end{bmatrix} \quad (31)$$

$$\begin{aligned} W_2 &:= \begin{bmatrix} I_m & 0 & \dots & 0 \end{bmatrix} \\ V_2 &:= \begin{bmatrix} R_0 & R_1 & \dots & R_{n-1} \end{bmatrix} \end{aligned} \quad (32)$$

and

$$H := \begin{bmatrix} R_1 & R_2 & \cdots & r_n \\ R_2 & & & \\ \vdots & & & \vdots \\ R_n & R_{n+1} & \cdots & R_{2n-1} \end{bmatrix} \quad (33)$$

we have the following

Theorem 3.1: Let Q be a $m \times p$ rational function whose set of poles does not intersect $\sigma(A_1) \cup \sigma(A_2)$. Then $Q = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ has a representation (28) if and only if there exist Y_1, Y_2 such that

$$[Y_1, W_1] \begin{bmatrix} A & B \\ C & D \end{bmatrix} = [\mathcal{A}_1 Y_1, V_1] \quad (34)$$

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Y_2 \\ W_2 \end{bmatrix} = \begin{bmatrix} Y_2 \mathcal{A}_2 \\ V_2 \end{bmatrix} \quad (35)$$

and

$$Y_1 Y_2 = -H \quad (36)$$

Proof. Observe first that the following identity, together with $Q(s_0) = R_0$, is equivalent to (28)

$$H = \int_\Gamma (sI - \mathcal{A}_1)^{-1} W_1 Q(s) W_2 (sI - \mathcal{A}_2)^{-1} ds \quad (37)$$

where Γ is any closed curve containing $\sigma(\mathcal{A}_1) \cup \sigma(\mathcal{A}_2)$ but not the poles of Q. In fact,

$$(sI - \mathcal{A}_1)^{-1} = \begin{bmatrix} I_p(s-s_0)^{-1} & & & \\ I_p(s-s_0)^{-2} & I_p(s-s_0)^{-1} & & \\ \vdots & & \ddots & \\ I_p(s-s_0)^{-n} & I_p(s-s_0)^{n-1} & \cdots & I_p(s-s_0)^{-1} \end{bmatrix}$$

and

$$(sI - \mathcal{A}_2)^{-1} = \begin{bmatrix} I_m(s-s_0)^{-1} & I_m(s-s_0)^{-2} & \cdots & I_m(s-s_0)^{-n} \\ & I_m(s-s_0)^{-1} & & I_m(s-s_0)^{n-1} \\ & & \ddots & \vdots \\ & & & I_m(s-s_0)^{-1} \end{bmatrix}$$

Thus

$$\begin{aligned} &(sI - \mathcal{A}_1)^{-1} W_1 Q(s) W_2 (sI - \mathcal{A}_2)^{-1} = \\ &\begin{bmatrix} I_p(s-s_0)^{-1} \\ I_p(s-s_0)^{-2} \\ \vdots \\ I_p(s-s_0)^{-n} \end{bmatrix} Q(s) \\ &\cdot [I_m(s-s_0)^{-1}, I_m(s-s_0)^{-2}, \cdots, I_m(s-s_0)^{-n}] \end{aligned}$$

and therefore, if Γ is any closed curve around s_0 and not containing the poles of Q,

$$\begin{aligned} &\frac{1}{2\pi i} \int_\Gamma (sI - \mathcal{A}_1)^{-1} W_1 Q(s) W_2 (sI - \mathcal{A}_2)^{-1} ds \\ &= \begin{bmatrix} R_1 & R_2 & \cdots & R_n \\ R_2 & & & \\ \vdots & & & \vdots \\ R_n & R_{n+1} & \cdots & R_{2n-1} \end{bmatrix} = H \end{aligned}$$

as claimed. Now, if $Q = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ satisfies (28), we know from Theorem 2.1 that there exist matrices Y_1, Y_2 satisfying (34) and (35), respectively. In view of (35), it is

$$[Q(s)W_2 - V_2](sI - \mathcal{A}_2)^{-1} = -C(sI - A)^{-1} Y_2$$

which is analytic in $\sigma(\mathcal{A}_2)$. Since Q also satisfies (37), we can write, using (25)

$$\begin{aligned} H &= \frac{1}{2\pi i} \int_\Gamma (sI - \mathcal{A}_1)^{-1} W_1 Q(s) W_2 (sI - \mathcal{A}_2)^{-1} ds \\ &= \frac{1}{2\pi i} \int_\Gamma (sI - \mathcal{A}_1)^{-1} W_1 [Q(s) W_2 - V_2] (sI - \mathcal{A}_2)^{-1} ds \\ &= -\frac{1}{2\pi i} \int_\Gamma (sI - \mathcal{A}_1)^{-1} W_1 C (sI - A)^{-1} Y_1 ds \\ &\qquad \text{in view of (35)} \\ &= -\frac{1}{2\pi i} \int_\Gamma (sI - \mathcal{A}_2)^{-1} [-Y_2 A_2 + \mathcal{A}_2 Y_2] (sI - A)^{-1} Y_2 ds \\ &\qquad \text{in view of (34)} \\ &= -\frac{1}{2\pi i} \int_\Gamma [(sI - \mathcal{A}_2)^{-1} Y_1 Y_2 - Y_1 (sI - A)^{-1} Y_2] ds \\ &= -Y_1 Y_2 \end{aligned}$$

■

If $-H = Y_1 Y_2$ is a factorization of $-H$ with Y_1 and Y_2 of full column and row rank, respectively, we can write a realization of Q as

$$Q = \left[\begin{array}{c|c} A_Q & -W_1 D + V_1 \\ \hline (-DW_2 + V_2) Y_2^{-R} & D \end{array}\right] \quad (38)$$

where

$$A_Q = Y_1^{-L}\mathcal{A}_1 Y_1 - Y_1^{-L} W_1(-DW_2 + V_2)Y_2^{-R}$$

IV. CONNECTING WITH THE HO-KALMAN ALGORITHM

We want to connect now with the usual partial realization problem, where we have interpolations values at infinity

$$\begin{array}{ll} R_0 = \hat{C}\hat{B} + D & \\ R_n = \hat{C}\hat{A}^{n-1}\hat{B} & \text{for } n > 0 \end{array} \tag{39}$$

for an interpolating function Q. This second problem can, in general, be reduced to the one considered above. In fact, from (29) with $s_0 = 0$ and from (39) we have

$$R_0 = -CA^{-1}B + D = \hat{C}\hat{B} + D$$

and

$$R_n = -CA^{-n}B = \hat{C}\hat{A}^{n-1}\hat{B} \qquad \text{for } n > 0$$

so that, if A is invertible, setting $\hat{A} = A^{-1}, \hat{B} = -A^{-1}B, \hat{C} = C, \hat{D} = D$, we immediately obtain a solution to our problem. In fact, with the assumption that $s_0 = 0$ and $\mathcal{A}_1, \mathcal{A}_2, W_1, W_2$ are as in (30)-(32), it is clear that the solutions to equations (3) and (4) are the observability and controllability matrices for $(\hat{A}, \hat{C})$ and $(\hat{A}, \hat{B})$ multiplyed on the proper side by $\hat{A}$.

If A is invertible, we can modify the above formulas as follows: set

$$\hat{Y}_1 = Y_1 A \qquad \hat{Y}_2 = Y_2$$

Then

$$Y_1 Y_2 = \hat{Y}_1 \hat{A} \hat{Y}_2 \tag{40}$$

and (3) and (4) become:

$$-\mathcal{A}_1 \hat{Y}_1 \hat{A} + \hat{Y}_1 + W_1 \hat{C} = 0 \tag{41}$$

$$\hat{Y}_2 - \hat{A}\hat{Y}_2 \mathcal{A}_2 + \hat{B} W_2 = 0 \tag{42}$$

and

$$V_1 := \hat{Y}_1 \hat{B} + W_1 D \tag{43}$$

$$V_2 := \hat{C}\hat{Y}_2 + DW_2 \tag{44}$$

Since $\hat{Y}_1\hat{Y}_2$ and $\hat{Y}_1\hat{A}\hat{Y}_2$ are known, from any factorization of Y_1Y_2, using (40), (43) and (44), we easily obtain $\hat{A}, \hat{B}, \hat{C}$ and thus the interpolating Q. This holds in view of the invertibility of A and Theorem 2.1. Now, as the data are such that A tends towards a non invertible matrix, all the limits exist and are finite; thus the limiting Q is still interpolating the data. This is, not surprisingly, a variation of the Ho-Kalman realization algorithm (see [8]).

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