Sampled-data Cross-track Control for Underactuated Ships

Hitoshi Katayama

Abstract—The sampled-data cross-track control problem for an underactuated three degree-of-freedom ship is considered. A line-of-sight guidance algorithm is used to design surge and yaw control laws which make a ship track a desired straight-line reference trajectory while maintaining a desired nonzero constant forward speed. Then applying the nonlinear sampled-data control theory and the stability theory of parametrized discrete-time cascade interconnected systems, it is shown that sampled-data cross-track is achieved by the designed control laws. Simulation results are also given to illustrate the design method.

I. INTRODUCTION

From the beginning of 20th century, the design of control systems for ships has been actively considered. Recent analysis and synthesis of the control problems for ships have been considered based on nonlinear continuous-time models and the design methods of continuous-time controllers have been mainly discussed (for details see [4] and references therein). For fully-actuated and underactuated ships, dynamic positioning control, trajectory tracking control, formation control and etc have been considered ([1]-[5]).

Practical and modern control systems usually use digital computers as discrete-time controllers with sampler (A/D converters) and zero-order holders (D/A converters) to control continuous-time systems. Such a control system involves both continuous-time and discrete-time signals in a continuous-time framework and is called a sampled-data system. Recently the framework to design controllers for nonlinear sampled-data systems based on discrete-time approximate models is proposed (for details see [10], [13], [14] and references therein). Several design methods have been also given to guarantee the stability of nonlinear sampled-data systems. Following this nonlinear sampled-data control theory, the design of semiglobally practically asymptotically (SPA) stabilizing state feedback and output feedback controllers for dynamically positioned ships has been considered ([6]-[8]).

In this paper we consider sampled-data cross-track control for underactuated three degree-of-freedom (3DOF) ships, which have only two, control inputs in surge and yaw. We introduce a straight-line as a reference trajectory between two successive way-points and a reference nonzero forward speed of a ship. Then the control objective is to design discrete-time controllers which make a ship track a desired straight-line trajectory while maintaining a desired nonzero constant forward speed in the continuous-time sense when the control input is given by the designed discrete-time controller with a zero-order hold. To achieve this control objective, we assume that the x-axis of the inertia reference coordinate system is a desired straight-line and we introduce a line-of-sight (LOS) guidance algorithm as in the continuous-time cross-track control of underactuated ships [2]. Then the y-position of a ship is a cross-track error and the LOS guidance algorithm or the LOS angle is given by the cross-track error. We can also define the tracking errors of the surge velocity and the yaw angle. First we consider two tracking error dynamics in surge and yaw and their Euler approximate models. We consider yaw and surge control, independently and we design globally asymptotically (GA) stabilizing state feedback laws for each Euler approximate model. Next we consider the Euler approximate models of a cross-track error dynamics and the tracking error dynamics in yaw and surge. Since these Euler approximate models can be rewritten as parametrized discrete-time cascade interconnection and the Euler approximate models in yaw and surge are GA stabilized by the designed state feedback laws in the discrete-time sense, by Nesic and Loria [12] we can show that the Euler approximate model of a cross-track error dynamics is also GA stabilized. Then by the results in [10], we can show that the SPA stability for the cascade interconnection of the continuous-time cross-track error dynamics and the tracking error dynamics in yaw and surge is achieved. Hence the designed discrete-time state feedback laws based on the Euler approximate models and the LOS guidance algorithm achieve sampled-data cross-track control of a ship in the continuous-time SPA stable sense. A numerical example is given to illustrate the design method.

This paper is organized as follows. In Section II we summarize a framework of nonlinear sampled-data control system designs and the stability of parametrized discrete-time cascade interconnected systems. In Section III we consider sampled-data cross-track control for underactuated 3DOF ships. In Section IV we give a numerical example to illustrate the proposed design method. In Section V we give a conclusion.

Notation: Let \( \mathbb{R} \) and \( \mathbb{R}_{\geq 0} \) be the sets of real numbers and nonnegative real numbers, respectively. Let \( \| \mathbf{x} \| \) be the norm of a vector \( \mathbf{x} \) given by \( \| \mathbf{x} \| = \sqrt{x^T x} \). A function \( \alpha \) is of class \( K \) if it is continuous, zero at zero and strictly increasing. It is of class \( K_\infty \) if it is of class \( K \) and unbounded. A function \( \beta: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) is of class \( KL \) if for any fixed \( t \geq 0 \), the function \( \beta(s,t) \) is of class \( K \) and for each fixed \( s \geq 0 \) the function \( \beta(s,\cdot) \) is decreasing to zero as its argument tends to infinity [9]. A function \( \gamma: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) is of class \( N \) if \( \gamma(\cdot) \) is continuous...
II. PRELIMINARY RESULTS

A. Framework of Nonlinear Sampled-data Control System Design

Here we summarize a framework for the design of nonlinear sampled-data systems by Euler approximate models. For details see [10], [13] and [14].

Consider the nonlinear continuous-time system

\[ \dot{x}_c = f(x_c, u_c), \quad x_c(0) = x_0 \]  

(1)

where \( x_c \in \mathbb{R}^n \) is the state, \( u_c \in \mathbb{R}^m \) is the control input realized through a zero-order hold, i.e., \( u_c(t) = u(k) \) for any \( t \in [kT, (k+1)T) \) and \( T > 0 \) is a sampling period. Here we assume that for each initial condition and each constant control, there exists a unique solution of (1) defined on some bounded interval of the form \( [0, \tau] \). We also assume that the sampling period is a design parameter and can be assigned arbitrarily.

Let \( x_c(k) = x_c(kT) \). Then the difference equations corresponding to the exact discrete-time model and the Euler approximate model of (1) are given by

\[ x_c(k+1) = F_T(x_c(k), u(k)), \quad x_c(0) = x_0, \]  

(2)

\[ \xi(k+1) = F_T^{\text{Euler}}(\xi(k), u(k)), \quad \xi(0) = x_0 \]  

(3)

respectively, where

\[ F_T(x_c, u(k)) = x_c(k) + \int_{kT}^{(k+1)T} f(x(s), u)ds \]

and \( F_T^{\text{Euler}}(\xi, u(k)) = \xi(k) + T f(\xi(k), u(k)) \). Note that the discrete-time models (2) and (3) are parametrized by \( T \).

To define stability of the parametrized discrete-time models, we first consider the following discrete-time system

\[ x(k+1) = F_T(x(k)), \quad x(0) = x_0 \]  

(4)

which represents a closed-loop system \( x(k+1) = F_T(x(k), u(k)) \) and a parameterized state feedback law \( u(k) = u_T(x(k)) \).

Definition 2.1: 1) The parametrized discrete-time system (4) is semiglobally practically asymptotically stable (SPAS) if there exists \( \beta \in \text{class}\, KL \) such that for any pair of strictly positive real numbers \( (D, d) \), there exists \( T^* > 0 \) such that for all \( x_0 \in \mathbb{R}^n \) with \( \| x_0 \| \leq D \) and \( T \in (0, T^*) \)

\[ \| x(k) \| \leq \beta(\| x_0 \|, kT) + d, \quad \forall k \geq 0. \]

2) The parametrized discrete-time system (4) is globally asymptotically stable (GAS) if there exists \( \beta \in \text{class}\, KL \) such that there exists \( T^* > 0 \) such that for all \( x_0 \in \mathbb{R}^n \) and \( T \in (0, T^*) \)

\[ \| x(k) \| \leq \beta(\| x_0 \|, kT), \quad \forall k \geq 0. \]

3) The parametrized discrete-time system (4) is globally bounded (GB) if there exist \( \alpha \in \text{class}\, K\infty \) and \( c \geq 0 \) such that there exists \( T^* > 0 \) such that for all \( x_0 \in \mathbb{R}^n \) and \( T \in (0, T^*) \)

\[ \| x(k) \| \leq \alpha(\| x_0 \|) + c. \]

4) The parametrized discrete-time system (4) is Lyapunov-GAS if there exist \( \alpha_1, \alpha_2 \in \text{class}\, K_{\infty} \), \( \alpha_3 \in \text{class}\, K \), \( L \in \text{class}\, N \), \( T^* > 0 \) and a continuous function \( V_T: \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) for each \( T \in (0, T^*) \) such that for all \( x, y \in \mathbb{R}^n \) and \( T \in (0, T^*) \)

\[ \alpha_1(\| x \|) \leq V_T(x) \leq \alpha_2(\| x \|), \]

(5)

\[ V_T(F_T(x)) - V_T(x) \leq -T \alpha_3(\| x \|), \]

(6)

\[ |V_T(x) - V_T(y)| \leq L \max \{\| x \|, \| y \| \} \times \| x - y \|. \]

(7)

Now we consider the design of stabilizing feedback laws \( u_T(x) \) for the exact discrete-time model (2). By [13] the Euler approximate model (3) with \( u(k) = u_T(\xi(k)) \) is one-step consistent with the exact discrete-time model (2) with \( u(k) = u_T(x_c(\xi)) \), i.e., for each compact set \( \mathcal{X} \subseteq \mathbb{R}^n \) there exist a function \( \rho \in \text{class}\, K_{\infty} \) and \( T^* > 0 \) such that for all \( x \in \mathcal{X} \) and \( T \in (0, T^*) \)

\[ \| F_T^e(x, u_T(x)) - F_T^{\text{Euler}}(x, u_T(x)) \| \leq T \rho(T). \]

Moreover, if \( f(x, u) \) and a parametrized state feedback law \( u_T(x) \) are locally Lipschitz for any \( T \in (0, T^*) \), the Euler approximate model (3) with \( u(k) = u_T(\xi(k)) \) is multi-step consistent with the exact discrete-time model (2) with \( u(k) = u_T(x_c(\xi)) \) [13], i.e., for each \( L \geq 0 \), \( l \geq 0 \), and each compact set \( \mathcal{X} \subseteq \mathbb{R}^n \), there exist a function \( \epsilon : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \cup \{0\} \) and \( T^* > 0 \) such that for all \( T \in (0, T^*), \{x, z \in \mathcal{X} : \| x - z \| < d \} \) implies

\[ \| F_T^e(x, u_T(x)) - F_T^{\text{Euler}}(x, u_T(x)) \| \leq \epsilon(d, T) \]  

and \( k \leq \frac{1}{\epsilon(d, T)} \) implies

\[ \epsilon(k, 0, T) = \epsilon(\cdots \epsilon(0, T), T), \ldots, T) \leq l. \]

Then by [13] we have the following stability result.

Theorem 2.1: If the Euler approximate model (3) with \( u(k) = u_T(\xi(k)) \) is multi-step consistent with the exact discrete-time model (2) with \( u(k) = u_T(x_c^{(x)}(k)) \) and the Euler approximate model (3) with \( u(k) = u_T(\xi(k)) \) is GAS, then the exact discrete-time model (2) with \( u(k) = u_T(x_c^{(x)}(k)) \) is SPAS.

Remark 2.1: If \( F_T^e \) is locally Lipschitz, then there exists \( T^* > 0 \) such that for any \( T \in (0, T^*), u_T \) which SPAS stabilizes the exact discrete-time model (2), EPA stabilizes (1), i.e., there exists \( \beta \in \text{class}\, KL \) such that for any strictly positive real numbers \( (D, d) \), there exists \( T^* > 0 \) such that for any \( T \in (0, T^*) \) and any \( x_0 \in \mathbb{R}^n \) satisfying \( \| x_0(k) \| \leq D \), a solution \( x_c(t) \) of the system \( \dot{x}_c = f(x_c, \tau_T(x_c(kT))) \) for any \( t \in [kT, (k+1)T) \) satisfies

\[ \| x_c(t) \| \leq \beta(\| x_0(0) \|, t) + d. \]

In this case we say that the system \( \dot{x}_c = f(x_c, \tau_T(x_c(kT))), \quad \forall t \in [kT, (k+1)T) \) is SPAS in the continuous-time sense ([13], [14]).
B. Stabilization of Parametrized Discrete-time Cascade Interconnected Systems

In this subsection we also summarize the stability of parametrized discrete-time cascade interconnected systems which are related to the SPA stability of the sampled-data cascaded interconnected systems. For details see [11] and [12].

Consider the parametrized discrete-time cascade interconnected system

\[
x(k + 1) = f_T(x(k), z(k)), \quad (8)
\]

\[
z(k + 1) = l_T(z(k)) \quad (9)
\]

which corresponds to a closed-loop system of the exact discrete-time model or the Euler approximate model of the system (8) and (9) is given by the following result ([11], [12]).

**Proposition 2.1:** Suppose that there exist \( \gamma_2 \in \text{class } N, \gamma_1, \gamma_3 \in \text{class } K_{\infty} \) and \( T^* > 0 \) such that for all \( \xi \in \mathbb{R}^{n+n} \) and \( T \in (0, T^*) \) we have

\[
\| f_T(x, z) \| \leq \gamma_1(\| \xi \|),
\]

\[
\| f_T(x, z) - f_T(x, 0) \| \leq T \gamma_2(\| x \|) \gamma_3(\| z \|)
\]

where \( \xi = [x^T \ z^T]^T \).

Then we have the following stability result of the parametrized discrete-time cascade system (8) and (9) ([12]).

**Theorem 2.2:** Assume A1. Then the system (8) and (9) is GAS if the following conditions hold

1. The system \( x(k + 1) = f_T(x(k), 0) \) is GAS.
2. The system (9) is GAS.
3. The system (8) and (9) is globally bounded.

The sufficient condition for the globally boundedness of the system (8) and (9) is given by the following result ([11], [12]).

**Proposition 2.1:** Consider the system (8) with input \( z \).

Suppose that there exist \( \tilde{\alpha}_1, \tilde{\alpha}_2, \varphi \in \text{class } K_{\infty}, \tilde{\gamma}_1, \tilde{\gamma}_2 \in \text{class } N, T^* > 0, c \geq 0 \) and for each \( T \in (0, T^*) \) there exists \( W_T : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \) such that for all \( x \in \mathbb{R}^n, z \in \mathbb{R}^n \) and \( T \in (0, T^*) \) we have

\[
\tilde{\alpha}_1(\| x \|) \leq W_T(x) \leq \tilde{\alpha}_2(\| x \|) + c,
\]

\[
W_T(f_T(x, z)) - W_T(x) \leq T \tilde{\gamma}_1(\| z \|) \varphi(W_T(x)) + T \tilde{\gamma}_2(\| z \|),
\]

\[
\int_1^\infty \frac{ds}{\varphi(s)} = \infty.
\]

If, furthermore, the solutions of (9) satisfy the summability condition

\[
T \sum_{k=0}^\infty \mu(\| z(k) \|) \leq \rho(\| z(0) \|)
\]

with some \( \rho \in \text{class } K_{\infty} \) and \( \mu(s) = \tilde{\gamma}_1(s) + \tilde{\gamma}_2(s)/\varphi(1) \), then the system (8) and (9) is globally bounded.

**Remark 2.2:** To design state feedback SPA stabilizing laws for the sampled-data cascade interconnected system (10) and (11) based on its Euler approximate model, we first check the assumption A1 and the condition 1) in Theorem 2.2 for the Euler approximate model of the form (8) and (9) and then we design a state feedback law that satisfies the conditions 2) and 3) in Theorem 2.2. Then if the designed state feedback GA stabilizing feedback law is locally Lipschitz, then by Theorem 2.1, the exact discrete-time model of the sampled-data system (10) and (11) with the designed state feedback law \( u(k) = u_T(z_c(kT)) \) is SPAS where \( z_c(k) = z_c(kT) \). Hence by Remark 2.1 the original sampled-data-cascade interconnected system (10) and (11) with \( u(t) = u_T(z_c(kT)), t \in [kT, (k + 1)T) \) is also SPAS for sufficiently small \( T > 0 \) since the exact discrete-time model is locally Lipschitz in general.

III. Sampled-data Cross-Track Control for Ships

A. Model of a Ship and a Problem Formulation

We first introduce notation to describe the equation of motion of a ship. Let \( [x_c \ y_c]^T \) and \( \psi_c \) be the inertial position and the yaw angle (orientation) of a ship, respectively in Cartesian coordinate system and let \( u_c, v_c \) and \( r_c \) be the linear velocities in surge, sway and the angular velocity in yaw, i.e., \( r_c = \dot{\psi}_c \), respectively, decomposed in the body-fixed coordinate system [4] (Figure 1). Let \( \nu_c = [u_c \ v_c \ r_c]^T \). Consider the 3DOF model of a ship ([2], [3]):

\[
\dot{x}_c = u_c \cos \psi_c - v_c \sin \psi_c,
\]

\[
\dot{y}_c = u_c \sin \psi_c + v_c \cos \psi_c,
\]

\[
\dot{\psi}_c = r_c,
\]

\[
\dot{u}_c = d_u(\nu_c) + \tau_{cu},
\]

\[
\dot{v}_c = d_v(\nu_c),
\]

\[
\dot{r}_c = d_r(\nu_c) + \tau_{cr}.
\]
where
\[
[d_u(\nu_c) \ d_v(\nu_c) \ d_r(\nu_c)]^T = -M^{-1}C(\nu_c)\nu_c - M^{-1}Du_c,
\]
\( \tau_{cu} \) and \( \tau_{cr} \) are a control force input and a yaw torque input, respectively and are realized through a zero-order hold, i.e.,
\[
\tau_{cu}(t) = \tau_u(k), \ \tau_{cr}(t) = \tau_r(k)
\]
for any \( t \in [kT, (k + 1)T] \) and \( T \) is a sampling period. Moreover
\[
M = \begin{bmatrix}
m_{11} & 0 & 0 \\
0 & m_{22} & m_{23} \\
0 & m_{23} & m_{33}
\end{bmatrix} > 0
\]
is the inertia matrix including hydrodynamic added inertia,
\[
D = \begin{bmatrix}
d_{11} & 0 & 0 \\
0 & d_{22} & d_{23} \\
0 & d_{32} & d_{33}
\end{bmatrix}
\]
with \( d_{ii} > 0, \ i = 1, 2, 3 \) is the damping matrix and
\[
C(\nu_c) = \begin{bmatrix}
0 & 0 & -m_{22}v_c - m_{23}r_c \\
m_{22}v_c + m_{23}r_c & -m_{11}u_c & 0
\end{bmatrix}
\]
is the Coriolis-centripetal matrix. Note that the model (16)-(21) expresses a class of underactuated ships where only 2 independent controls are available to control three degrees-of-freedom (3DOF) motion. Moreover we assume that the control input does not affect the sway motion directly.

For the surge and the sway velocities, we introduce the following natural assumption [2]:

**B1:** The surge velocity \( u_c(t) \) satisfies
\[
0 < U_{min} \leq u_c(t) \leq U_{max}, \ \forall t \geq 0
\]
(22)
where \( U_{min} \) and \( U_{max} \) are the maximum and the minimum surge velocities, respectively and for some \( C_u > 0 \) the sway velocity \( v_c(t) \) satisfies
\[
|v_c(t)| \leq C_u U_{max} |r_c(t)|, \quad \forall t \geq 0
\]
(23)
\[
|v_c(t)| \leq U_{max}, \quad \forall t \geq 0.
\]
(24)

The condition (22) is needed to make the system controllable in the sway direction and the conditions (23) and (24) are valid for most underactuated ships, since the hydrodynamic damping in the sway direction is usually much larger than the hydrodynamic damping in the surge direction. Moreover, (23) implies that if the angular velocity \( r_c \) converges to zero, then the sway velocity \( v_c \) also converges to zero.

Way-point guidance (or tracking) is one of the basic control problems of a ship. The route of a ship is given by a straight-line connecting two successive specified way-points (for details see [4]). We set the origin of the Cartesian coordinate system in the previous way-point and the \( x \)-axis pointing toward to the new way-point. The \( y \)-axis is chosen to complete the right-handed coordinate system (Figure 1). In this case, the \( y \)-position of a ship is a cross-track error and the \( x \)-axis corresponds to a desired straight-line trajectory. Then we want to find state feedback laws which make a ship track the \( x \)-axis, while maintaining a desired nonzero constant forward speed. Let \( \nu_c \in [U_{min}, U_{max}] \) be a desired forward speed and assume that the state of a ship at each sampling time, i.e., \((x_c(kT), y_c(kT), \psi_c(kT), v_c(kT))\), \( k = 0, 1, 2, \ldots \) is available to state feedback laws. Then the control objective is to design discrete-time state feedback laws which satisfy \((y_c(t), \psi_c(t), u_c(t) - \bar{u}_c, v_c(t), r_c(t)) \to 0 \) as \( t \to 0 \) for the obtained closed-loop system in the continuous-time SPA sense.

**B. Sampled-data Cross-track Control**

In this subsection we consider sampled-data cross-track control based on the line-of-sight (LOS) guidance algorithm and the nonlinear sampled-data control system design theory. The LOS guidance is often used for path control of ships. We use the LOS guidance algorithm to achieve the control objective. Since the dynamics (17) contains no control inputs, we must control the surge speed \( u_c \) and the yaw angle \( \psi_c \) to make \( |y_c(t)| \) small. We pick a point that lies a constant distance \( \Delta > 0 \) ahead of a ship, along the trajectory. The line of sight is the line joining a ship and the selected point. The angle describing the orientation of the line of sight is called a LOS angle and \( \Delta > 0 \) is called the look-ahead distance (Figure 1). The LOS angle is given by
\[
\bar{\psi}_c(t) = \tan^{-1} \left( -\frac{y_c(t)}{\Delta} \right).
\]
(25)
In the following we first design a surge control law which makes the surge velocity \( u_c(t) \) track a desired constant speed \( \bar{u}_c \) and a yaw control law which makes the yaw angle \( \psi_c(t) \) track the LOS angle \( \bar{\psi}_c(t) \). Then we show that the cross-track error \( y_c(t) \) and the yaw angle \( \psi_c(t) \) converges to zero in the continuous-time SPA sense.

Let
\[
\tilde{u}_c(t) = u_c(t) - \bar{u}_c, \quad \tilde{\psi}_c(t) = \psi_c(t) - \bar{\psi}_c(t).
\]
Then using (18)-(21), we have
\[
\dot{\tilde{u}}_c = d_u(\nu_c) + \tau_{cu},
\]
\[
\dot{\tilde{\psi}}_c = r_c - \tilde{r}_c - k_\psi \tilde{\psi}_c = -k_\psi \tilde{\psi}_c + \tilde{r}_c,
\]
\[
\dot{\tilde{r}}_c = d_r(\nu_c) - \kappa(y_c, \psi_c, \nu_c) + \tau_{cr}
\]
(26)
(27)
(28)
where \( \tilde{r}_c = r_c - \bar{r}_c \),
\[
\tilde{r}_c = \tilde{r}_c + k_\psi \tilde{\psi}_c + \tilde{\psi}_c = -k_\psi \tilde{\psi}_c - \frac{\Delta}{\Delta^2 + y_c^2} \dot{y}_c
\]
(29)
and
\[
\kappa(y_c, \psi_c, \nu_c) = \tilde{r}_c - k_\psi \dot{\tilde{\psi}}_c - \frac{d}{dt} \left( \frac{\Delta}{\Delta^2 + y_c^2} \dot{y}_c \right).
\]
Note that \( \kappa(y_c, \psi_c, \nu_c) \) can be calculated exactly by (17)-(21) and \( k_\psi > 0 \) is a design parameter which is assigned later.

Let
\[
z_c = \frac{1}{2} y_c^2.
\]
By (23), (29) and the definition of $\tilde{c}$, we have
\[
|\nu_c| \leq C_v U_{\text{max}} (|\tilde{r}_c| + |\tilde{f}_c|) \\
\leq C_v U_{\text{max}} (|\tilde{r}_c| + k_\psi |\psi_c| + |\frac{\Delta}{\Delta^2 + y_c^2}| |\tilde{y}_c|) \\
\leq C_v U_{\text{max}} (|\tilde{r}_c| + k_\psi |\psi_c| + \frac{1}{\Delta} |\tilde{y}_c|)
\]
and by (17) we obtain
\[
\dot{z}_c = y_c \dot{\bar{c}}_c \\
\leq y_c u_c \sin \psi_c + |y_c| |v_c| \\
\leq y_c u_c \sin \psi_c + C_v U_{\text{max}} |y_c| (|\tilde{r}_c| + k_\psi |\psi_c|) \\
+ \frac{1}{\Delta} C_v U_{\text{max}} |\tilde{z}_c|.
\]
Since
\[
u_c \sin \psi_c = \bar{u}_c \sin \tilde{\psi}_c + \bar{u}_c \sin (\tilde{\psi}_c + \tilde{\psi}_c) \\
+ \bar{u}_c \sin \tilde{\psi}_c \cos \psi_c + \tilde{u}_c (\cos \tilde{\psi}_c - 1) \sin \psi_c \\
= \bar{u}_c \sin \tilde{\psi}_c + \bar{u}_c \sin (\psi_c + \psi_c) \\
+ \tilde{u}_c \left\{ \cos \tilde{\psi}_c \sin \tilde{\psi}_c + \sin \tilde{\psi}_c \cos \tilde{\psi}_c - 1 \right\} \tilde{\psi}_c,
\]
we have
\[
\dot{z}_c \leq y_c \bar{u}_c \sin \tilde{\psi}_c + y_c \bar{u}_c \sin (\tilde{\psi}_c + \tilde{\psi}_c) \\
+ y_c \bar{u}_c \left\{ \cos \tilde{\psi}_c \sin \tilde{\psi}_c + \sin \tilde{\psi}_c \cos \tilde{\psi}_c - 1 \right\} \tilde{\psi}_c \\
+ C_v U_{\text{max}} |\tilde{z}_c| (|\tilde{r}_c| + k_\psi |\psi_c|) \\
+ \frac{1}{\Delta} C_v U_{\text{max}} |\tilde{z}_c|
\]
where $\zeta = [\bar{u}_c \tilde{\psi}_c \tilde{r}_c]^T$ and
\[
q(\zeta) = |\bar{u}_c| + (2\bar{u}_c + k_\psi C_v U_{\text{max}}) |\tilde{\psi}_c| + C_v U_{\text{max}} |\tilde{r}_c|.
\]
Using
\[
\sin \tilde{\psi}_c = \frac{y_c}{\sqrt{\Delta^2 + y_c^2}},
\]
we obtain
\[
\dot{z}_c \leq \frac{\bar{u}_c}{\sqrt{\Delta^2 + y_c^2}} y_c^2 + |y_c| |q(\zeta)| + \frac{1}{\Delta} C_v U_{\text{max}} |\tilde{z}_c|
\]
and
\[
\left\{ 1 - \frac{C_v U_{\text{max}}}{\Delta} \right\} \dot{z}_c \leq -\frac{2\bar{u}_c}{\sqrt{\Delta^2 + 2z_c}} z_c + \sqrt{2z_c} q(\zeta)
\]
where $\text{sgn}(z)$ is a sign function. Now introduce the following assumption [2]:

**B2:** $U_{\text{max}} < \Delta \bar{c}_v$.

Let
\[
\Pi_1 = \left( 1 + \frac{C_v U_{\text{max}}}{\Delta} \right)^{-1}, \quad \Pi_2 = \left( 1 - \frac{C_v U_{\text{max}}}{\Delta} \right)^{-1}.
\]
Then $\Pi_2 > \Pi_1 > 0$ and we obtain
\[
\dot{z}_c \leq -\frac{2\bar{u}_c \Pi_1}{\sqrt{\Delta^2 + 2z_c}} z_c + \Pi_2 \sqrt{2z_c} q(\zeta) \quad (30)
\]
By the comparison lemma [9], a solution $\eta_c(t)$ of the differential equation
\[
\dot{\eta}_c = -\frac{2\bar{u}_c \Pi_1}{\sqrt{\Delta^2 + 2\eta_c}} \eta_c + \Pi_2 \sqrt{2\eta_c} q(\zeta) \quad (31)
\]
satisfies $0 \leq \eta_c(t) \leq \eta_c(t)$ if $0 \leq \eta_c(0) \leq \eta_c(t)$. Hence if control inputs $\tau_{cu}$ and $\tau_{cr}$ stabilize the system (26)-(28) and (31) with $\eta_c(0) = \gamma_0^2$, then $\tau_{cu}$ and $\tau_{cr}$ stabilize the system the system (17) and (26)-(28) with $y_c(0) = \gamma_0$, since $z_c = \frac{1}{2} \gamma_0^2$ and $0 \leq \eta_c(t) \leq \eta_c(t)$, $\forall t \geq 0$. Summing up we have the following result.

**Lemma 3.1:** Assume B1 and B2. Then if the control inputs $(\tau_{cu}(t), \tau_{cr}(t)) = (\tau_u(k), \tau_r(k))$, $\forall t \in [kT, (k+1)T)$ SPA stabilizes the system (26)-(28) and (31) with $\eta_c(0) = \gamma_0^2$, then $(\tau_{cu}(t), \tau_{cr}(t)) = (\tau_u(k), \tau_r(k), \forall t \in [kT, (k+1)T)$ STA stabilizes the system (17) and (26)-(28) with $y_c(0) = \gamma_0$.

**C. Design of surge and yaw control laws**

Since $\tau_{cu}(t) = \tau_u(k)$ and $\tau_{cr}(t) = \tau_r(k)$ for any $t \in [kT, (k+1)T)$, the Euler approximate model of the system (31), (26)-(28) is given by
\[
\eta(k+1) = f_T(\eta(k), \zeta(k)), \quad (32)
\]
\[
\ddot{u}(k+1) = \ddot{u}(k) + T \left[ d_u(\nu(k)) + \tau_u(k) \right], \quad (33)
\]
\[
\ddot{v}(k+1) = \left( 1 - T k_\nu \right) \ddot{v}(k) + T \ddot{r}(k), \quad (34)
\]
\[
\ddot{r}(k+1) = \ddot{r}(k) + T \left[ d_r(\nu(k)) \right], \quad \kappa(y(k), \psi(k), \nu(k)) + \tau_r(k) \quad (35)
\]
where $\zeta = [\ddot{u} \ddot{\psi} \ddot{r}]^T$
\[
f_T(\eta, \zeta) = F_T(\eta) + TG_T(\eta, \zeta),
\]
\[
F_T(\eta) = \left( 1 - T \frac{2\bar{u}_c \Pi_1}{\sqrt{\Delta^2 + 2\eta_c}} \right) \eta_c,
\]
\[
G_T(\eta, \zeta) = G_{1T}(\eta, \zeta) G_{2T}(\zeta),
\]
\[
G_{1T} = \sqrt{2\eta_c^2} \zeta, \quad \kappa(y(k), \psi(k), \nu(k)) + \tau_r(k) \quad (35)
\]
Let $T^* > 0$ be a maximal admissible sampling period and
\[
\tau_u(k) = -d_u(\nu(k)) - k_u \ddot{u}(k)
\]
for the system (33). Then we have the following result.

**Lemma 3.2:** If $0 < k_u \leq \frac{1}{T}$, then the surge control law (36) globally exponentially stabilizes the Euler approximate model (33) for any $T \in (0, T^*)$.

**Proof.** The closed-loop system (33) with (36) is given by
\[
\ddot{u}(k+1) = \left( 1 - T k_\nu \right) \ddot{u}(k)
\]
and we have $|1 - T k_\nu| < 1$ for any $T \in (0, T^*)$. Hence we have the assertion. \[\square\]

For the system (34) and (35) let
\[
\tau_r(k) = -d_r(\nu(k)) + \kappa(y(k), \psi(k), \nu(k)) - k_r \ddot{r}(k)
\]
Then we have the following result.

**Lemma 3.3:** If $0 < k_{\psi} \leq \frac{2}{\pi}$ and $0 < k_r \leq \frac{2}{\pi}$, then the yaw control law (38) globally exponentially stabilizes the Euler approximate model (34) and (35) for any $T \in (0, T^*)$.

**Proof.** The closed-loop system (34), (35) and (38) is given by

$$
\begin{bmatrix}
\dot{\psi} \\
\dot{\rho}
\end{bmatrix}
(k+1) =
\begin{bmatrix}
1 - Tk_{\psi} & 0 \\
0 & 1 - Tk_r
\end{bmatrix}
\begin{bmatrix}
\dot{\psi} \\
\dot{\rho}
\end{bmatrix}
(k)
$$

and we have $|1 - Tk_{\psi}| < 1$ and $|1 - Tk_r| < 1$ for any $T \in (0, T^*)$. Hence we have the assertion.

**D. SPA Stability of Sampled-data Cascade Interconnected Systems**

Now we shall show that the designed surge and yaw control laws SPA stabilize the sampled-data cascade interconnected system (26)-(28) and (31) by Theorems 2.1 and 2.2. First we shall show the global asymptotic stability of the system

$$
\eta(k+1) = f_T(\eta(k), \zeta(k)),
$$

$$
\zeta(k+1) =
\begin{bmatrix}
1 - Tk_u & 0 & 0 \\
0 & 1 - Tk_{\psi} & T \\
0 & 0 & 1 - Tk_r
\end{bmatrix}
\zeta(k)
$$

which is the closed-loop system (32)-(35) with the control laws (36) and (38).

**Lemma 3.4:** If $0 < k_u \leq \frac{2}{\pi}$, $0 < k_{\psi} \leq \frac{2}{\pi}$ and $0 < k_r \leq \frac{2}{\pi}$, then the closed-loop system (39) and (40) is GAS for any $T \in (0, T^*)$.

**Proof.** It is enough to show that the assumption $A_1$ and the conditions 1)-3) in Theorem 2.2 are satisfied for the system (39) and (40).

First we shall show that the assumption $A_1$ is satisfied for the Euler approximate model (39). In fact

$$
|f_T(\eta, \zeta)| \leq |f_T(\eta)| + |T|G_T(\eta, \zeta) |
\leq |\eta| + T\Pi_2\sqrt{2\eta q}(\zeta)
\leq \gamma_1(\| \eta \|)
$$

and

$$
|f_T(\eta, \zeta) - f_T(\eta, 0)| = T\Pi_2\sqrt{2\eta q}(\zeta) \leq \gamma_2(\| \zeta \|) \gamma_3(\| \zeta \|)
$$

for any $\tilde{\eta} = [\eta^T \ \ \zeta^T]^T$ where

$$
\gamma_3(s) = \Theta s, \\
\gamma_2(s) = \Pi_2 \sqrt{2s}, \\
\gamma_1(s) = s + T^* \gamma_2(s) \gamma_3(s), \\
\Theta = 1 + 2u_d + (1 + k_{\psi})C_u U_{\text{max}}.
$$

Obviously, $\gamma_1, \gamma_3 \in \text{class } K_{\infty}$, $\gamma_2 \in \text{class } N$ and hence $A_1$ is satisfied for the system (39).

Next we shall show that the condition 1) in Theorem 2.2 is satisfied. Let $T^* = \frac{\Delta}{2\bar{u}_c \Pi_1}$ for the system $\eta(k+1) = F_T(\eta(k))$. Then for any $T \in (0, T^*)$, we have

$$
1 - T^* \frac{2\bar{u}_c \Pi_1}{\sqrt{\Delta^2 + 2\eta}} > 1 - \frac{\Delta}{2\bar{u}_d \Pi_1 \sqrt{\Delta^2 + 2\eta}} = 1 - \frac{\Delta}{\sqrt{\Delta^2 + 2\eta}} > 0
$$

and

$$
T \frac{2\bar{u}_c \Pi_1}{\sqrt{\Delta^2 + 2\eta}} < T^* \frac{2\bar{u}_c \Pi_1}{\sqrt{\Delta^2 + 2\eta}} < 1.
$$

Hence $\eta(k+1) = F_T(\eta(k))$ is GAS (GES) for any $T \in (0, T^*)$. Note also that $\eta(k) \geq 0$, $\bar{v}k \geq 0$ for any $\eta(0) \geq 0$. Let $V_T(\eta) = \frac{1}{2}\eta^2$. Then we have

$$
V_T(\eta(k+1)) - V_T(\eta(k)) = -T \frac{2\bar{u}_c \Pi_1}{\sqrt{\Delta^2 + 2\eta}} \eta^2 + \frac{1}{2} T^2 \left( \frac{2\bar{u}_c \Pi_1}{\sqrt{\Delta^2 + 2\eta}} \right)^2 \eta^2
\leq -T \frac{2\bar{u}_c \Pi_1}{\sqrt{\Delta^2 + 2\eta}} \eta^2 + \frac{1}{2} T^* \left( \frac{2\bar{u}_c \Pi_1}{\sqrt{\Delta^2 + 2\eta}} \right) \Delta \eta^2
= -T \alpha(\eta)
$$

where

$$
\alpha(s) = \left[ 1 - \frac{T^*}{2} \frac{2\bar{u}_c \Pi_1}{\Delta} \right] \frac{2\bar{u}_c \Pi_1}{\sqrt{\Delta^2 + 2s^2}}.
$$

Since

$$
1 > 1 - \frac{1}{2} T^* \frac{2\bar{u}_c \Pi_1}{\Delta} > \frac{1}{2},
$$

we obtain $\alpha(s) \in \text{class } K_{\infty}$ and we also have

$$
|V_T(\eta) - V_T(s)| = \frac{1}{2} \left| \eta^2 - s^2 \right|
\leq \frac{1}{2} \left( |\eta| + |s| \right) |\eta - s|
\leq \max \left| \left| \eta \right|, |s| \right| |\eta - s|.
$$

Hence the system $\eta(k+1) = F_T(\eta(k))$ is Lyapunov GAS (GES) for any $T \in (0, T^*)$.

By the forms of the surge and yaw control laws, it is obvious that the condition 2) in Theorem 2.2 is satisfied and the designed control laws are locally Lipschitz. Using Proposition 2.1, we shall show that the condition 3) is satisfied. Let

$$
W_T(\eta) = |\eta|.
$$
Then the condition (12) is satisfied where \( \hat{\alpha}_1(s) = \hat{\alpha}_2(s) = s \) and \( c = 0 \). We also have

\[
W_T(f_T(\eta, \zeta)) - W_T(\eta)
\]

\[
= |F_T(\eta) + TG_T(\eta, \zeta)| - |\eta|
\]

\[
\leq |F_T(\eta) - \eta + TG_1T(\eta, \zeta)G_{2T}(\zeta)|
\]

\[
= T\frac{2\bar{u}_c\Pi_1}{\Delta}[|\eta| + |T|G_{1T}(\eta, \zeta)||G_{2T}(\zeta)|]
\]

\[
\leq T\frac{2\bar{u}_c\Pi_1}{\Delta}[|\eta| + T\frac{1}{2}(G_{1T}^2(\eta, \zeta) + G_{2T}^2(\zeta))]
\]

\[
= T\left\{ \frac{2\bar{u}_c\Pi_1}{\Delta} + q(\zeta) \right\}[|\eta| + T\frac{1}{2}\Pi_2^2q(\zeta)]
\]

\[
\leq T\left\{ \frac{2\bar{u}_c\Pi_1}{\Delta} + \Theta \parallel \| \zeta \| \right\} W_T(\eta) + T\frac{1}{2}\Pi_2^2\Theta \parallel \zeta \parallel .
\]

Let

\[
\varphi(s) = s,
\]

\[
\hat{\gamma}_1(s) = \frac{2\bar{u}_c\Pi_1}{\Delta} + \Theta s,
\]

\[
\hat{\gamma}_2(s) = \frac{1}{2}\Pi_2^2\Theta s.
\]

Then \( \varphi \in \text{class } K_{\infty} \), \( \hat{\gamma}_1 \), \( \hat{\gamma}_2 \in \text{class } N \) and we obtain (13). Since \( \varphi(s) = s \), (14) is satisfied. Note that

\[
\mu(s) = \hat{\gamma}_1(s) + \frac{\hat{\gamma}_2(s)}{\varphi(1)} = \frac{2\bar{u}_c\Pi_1}{\Delta} + \left( 1 + \frac{1}{2}\Pi_2^2 \right) \Theta s
\]

and (15) is satisfied since the system (40) is linear and GES. Hence by Proposition 2.1, the condition 3) in Theorem 2.2 is satisfied.

Consequently by Theorem 2.2, the closed-loop system (39) and (40) is GAS for any \( T \in (0, T^*) \).

Since the system (16)-(21) and the control laws (36) and (38) are locally Lipschitz, by Lemmas 3.1, 3.4 and Theorem 2.1, we have the following result.

**Theorem 3.1:** Assume B1 and B2. Then there exists \( T^* > 0 \) such that for any \( T \in (0, T^*) \) the surge and yaw control laws

\[
\tau_u(k) = -d_u(\nu_c(kT)) - k_u\bar{u}_c(kT),
\]

(41)

\[
\tau_v(k) = -d_r(\nu_c(kT)) + \kappa(\psi_c(kT), \nu_c(kT)) - k_r\bar{r}_c(kT)
\]

(42)

with \( 0 < k_u \leq \frac{1}{\bar{u}_c}, 0 < k_{\psi} \leq \frac{2\Delta}{\bar{u}_c(\psi)} \) and \( 0 < k_r \leq \frac{2\Delta}{\bar{u}_c(\psi)} \) satisfy \( (y_c(t), \psi_c(t), u_c(t) - \bar{u}_c, \nu_c(t), r_c(t)) \to 0 \) as \( t \to \infty \) in the continuous-time SPA sense for the sampled-data system (16)-(21).

**IV. Numerical Example**

Consider the system (16)-(21) for a supply vessel [5] where

\[
M = \begin{bmatrix}
7.22 & 0 & 0 \\
0 & 12.1 & -36.3 \\
0 & -36.3 & 4750
\end{bmatrix} \times 10^6,
\]

\[
D = \begin{bmatrix}
95070 & 0 & 0 \\
0 & 4.34 \times 10^6 & -2.47 \times 10^6 \\
0 & -1.88 \times 10^7 & 7.57 \times 10^8
\end{bmatrix}.
\]

Let Cartesian coordinate system be the Earth-fixed coordinate system where \( x_c \) and \( y_c \) are the North and East positions of a ship, respectively. For this system we set \( \bar{u}_c = 2 \text{ m/s}, \Delta = 20, U_{\text{max}} = 3 \text{ m/s} \) and we assume \( C_v = 1 \) for simplicity of discussion. Then the assumption B2 is satisfied. We set a sampling period \( T = 0.2 \text{ s} \) and choose

\[
k_u = 3, \quad k_{\psi} = k_r = 2.
\]

for the surge and yaw control laws (41) and (42). Let

\[
x_c(0) = 0, \quad y_c(0) = 5, \quad \psi_c(0) = 0,
\]

\[
u_c(0) = 0.2, \quad v_c(0) = 0, \quad r_c(0) = 0.
\]

be an initial condition of the ship. The solid line and symbols like a ship in Figure 2 express the trajectory of the position and the attitude of the ship at \( t = 0, 5, 10, \ldots, 65 \text{ s} \) respectively in the Earth-fixed coordinate system. Figure 3 shows the time responses of the surge velocity (a blue solid line) and the sway velocity (a black dotted line) and Figure 4 shows the time responses of the angular velocity in yaw. As we see Figures 2-4, the designed surge and yaw control laws achieve the control objective, in fact we have \( (y_c(65), \psi_c(65), u_c(65), v_c(65), r_c(65)) = (0.0016, -0.0047, 1.9826, -0.001, 0) \).

**V. Conclusion**

In this paper we have considered sampled-data cross-track control problem for underactuated 3DOF ships. We have used a LOS guidance algorithm to design control laws which make a ship track a desired straight-line reference trajectory while maintaining a desired nonzero constant forward speed. Using the Euler approximate models of the error dynamics in surge and yaw, we have designed discrete-time surge and yaw control laws, independently. Then applying the nonlinear sampled-data control theory and the stability theory of parametrized discrete-time cascade interconnected systems, we have shown that sampled-data cross track control is achieved by the designed control laws. Simulation results have been also given to illustrate the design method.
Fig. 3. Time responses of the surge and sway velocities of the ships (surge velocity: blue solid line, sway velocity: black dotted line - -)

Fig. 4. Time responses of the angular velocity of yaw

REFERENCES


