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Efficient Model Predictive Control for Linear Periodic Systems

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Abstract— **This paper proposes a novel model predictive control scheme for the stabilization of constrained linear periodically time-varying systems. The results are based on an existing Model Predictive Control scheme for uncertain linear systems using linear matrix inequalities. A pre-determined periodic feedback control law is used in combination with superimposed free control moves as additional degrees of freedom. Only the additional free control moves are calculated online taking advantage of pre-computed periodic invariant sets. Two simple algorithms are presented for calculating offline ellipsoidal or polyhedral periodic invariant sets. Since only a small number of free control moves is calculated online by solving a convex optimization problem after each time period, the computational cost can be reduced significantly compared to existing schemes.**

I. Introduction

Linear periodic dynamics are found in many real technical and nontechnical systems. Examples of periodic systems are given by the rotor motion of wind-turbines [13] and helicopters [1], compressors of jet engines [14], input-multirate systems [15] and models of economical systems [10].

Several different approaches for the control of linear periodic systems have been published. Output feedback stabilization via a Riccati equation approach is addressed in [9]. Conditions based on Linear Matrix Inequalities (LMIs) for state feedback and output feedback of unconstrained linear periodic systems are given in [12]. The issue of robustness is considered in [3]. Model Predictive Control (MPC) schemes for linear periodic systems are suggested in e.g. [5, 11, 15–17].

In this paper, we propose a scheme for linear periodic systems based on MPC. The primary advantage of MPC is its capability to deal with input and state constraints in a natural fashion. Its main disadvantage is the computational demand of the required online solution of a finite horizon optimal control problem. Much effort has been made to find problem formulations that allow a reduction of the online computations. For instance, [18, 19] provide LMI-based MPC schemes for uncertain linear systems whereas [4] considers periodic systems. In all of those schemes, the formulation in terms of LMIs allows the use of efficient solvers for convex optimization problems which makes these schemes computationally attractive [7].

The main goal of this work is to reduce the computational load compared to the existing scheme in [4]. The main idea is to use a pre-determined feedback law and calculate online superimposed control moves using the current state information along the lines of [19]. An extended state composed of the original system state and future free control moves is introduced. The concept of pre-determined periodic invariant sets in the extended state space plays a crucial part in order to guarantee properties such as feasibility and constraint satisfaction. The invariance property yields feasibility of future optimization problems provided that the extended state is contained within the periodic invariant sets. Furthermore, constraint satisfaction is ensured under the assumption that these invariant sets are calculated to be subsets of the constraint sets. As in [23], stability and optimality are achieved by the constraint of calculating the superimposed control moves such that the upper bound of an infinite horizon objective is minimized. There are essentially two ways for calculating these sets, namely calculation of ellipsoidal invariant sets [19] or polyhedral invariant sets [20–25]. In this paper, we investigate and compare both approaches for periodic systems.

The remainder of the paper is organized as follows: In Section II, the considered control problem is presented. Necessary assumptions are given in Section III. In Section IV, two algorithms for the calculation of periodic invariant sets are explained. The main result, a novel stabilizing MPC controller design is introduced in Section V. The main properties of the presented algorithms and the MPC scheme are illustrated via numerical examples in Section VI. Section VII concludes with a brief summary.

II. Problem Setup

Consider the linear N-periodic system of the form

$$x_{k+1} = A_k x_k + B_k u_k \tag{1}$$

with initial condition $x_0 = \bar{x}_0$. In (1), $x_k \in \mathbb{R}^n$ denotes the system state, $u_k \in \mathbb{R}^m$ is the control input, $k \geq 0$ is the discrete time variable, and $A_{k+N} = A_k \in \mathbb{R}^{n\times n}$, $B_{k+N} = B_k \in \mathbb{R}^{n\times m}$ are periodic matrices with time period N.

Furthermore, we assume the state and the input of the system to be bounded by polyhedral constraint sets

$$\Psi_x x_k \leq 1, \quad \Psi_u u_k \leq 1 \tag{2}$$

at each time instant k, in which $\Psi_x \in \mathbb{R}^{c_x\times n}$ and $\Psi_u \in \mathbb{R}^{c_u\times m}$ are approriate matrices in order to define c_x constraints on the state and c_u constraints on the input.

In the following, the goal is to design a model predictive controller which asymptotically stabilizes the origin of system (1) such that the state and input constraints defined above are satisfied at every time instant k.

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III. Preliminaries

A. Recalculation Scheduling

Throughout this work, we assume recalculation of the predictive control law at the beginning of each period, i.e. at discrete times $0, N, 2N, ...$ Therefore, the solution for the optimal input obtained at time instant k is applied to the system for all $k+i$ with $i = 0, 1, ..., N-1$. Note that recalculation after each discrete time step is also possible as in [26], but the details are omitted in this work for the sake of simplicity.

B. Control Parametrization

We consider a control parametrization similar to the one used in [19] as

$$u_{k+i|k} = \begin{cases} K_{k+i}x_{k+i} + c_{k+i|k} & i = 0, 1, ..., n_c - 1 \\ K_{k+i}x_{k+i} & i \geq n_c \end{cases} . \quad (3)$$

Herein $K_{k+i+N} = K_{k+i}$ denote N-periodic feedback matrices calculated offline such that the linear feedback $u_{k+i} = K_{k+i}x_{k+i}$ stabilizes system (1) in the absence of constraints. Furthermore, $c_{k+i|k} \in \mathbb{R}^m$ are n_c superimposed control moves calculated online after each period in order to improve performance and restore contraint satisfaction. Note that in contrast to the MPC scheme in [4], the feedback matrices K_{k+i} are calculated offline and therefore the computational demand is significantly reduced.

C. State Extension

We introduce the extended state $z \in \mathbb{R}^{n+mn_c}$

$$z_k = \begin{bmatrix} x_k^T & c_{k|k}^T & c_{k+1|k}^T & \cdots & c_{k+n_c-1|k}^T \end{bmatrix}^T \quad (4)$$

composed of the state x_k and the superimposed control moves $c_{k+i|k}$, $i = 0, 1, ..., n_c - 1$. Using the control parametrization from (3), the dynamics of system (1) and the superimposed control moves can be defined by the autonomous periodic system

$$z_{k+1} = \Phi_k z_k, \quad \Phi_k = \Phi_{k+N}, \quad (5)$$

in which

$$\Phi_k = \left[\begin{array}{c|c|c} A_k + B_k K_k & B_k & 0 \\ \hline 0 & 0 & I_{m(n_c-1)} \\ & 0 & 0 \end{array}\right] . \quad (6)$$

Here $I_{m(n_c-1)}$ denotes the $m(n_c-1) \times m(n_c-1)$ identity matrix. Using the control parametrization from (3), the input and state constraints in (2) can be rewritten as the following periodic polyhedral constraint sets

$$S_{C,k}^z = \left\{ z \mid \left[\begin{array}{c|c|c} \Psi_x & 0 & 0 \\ \hline \Psi_u K_k & \Psi_u & \end{array}\right] z \leq 1 \right\} . \quad (7)$$

Note the periodicity of the constraint sets $S_{C,k}^z = S_{C,k+N}^z$ because of the periodicity of the pre-determined feedback matrices $K_{k+N} = K_k$.

D. Upper Bound on the Infinite Horizon Cost

We consider the infinite horizon objective function

$$J_{\infty,k} = \sum_{i=0}^{\infty} x_{k+i|k}^T Q x_{k+i|k} + u_{k+i|k}^T R u_{k+i|k} . \quad (8)$$

Analogue to [23] and using control parametrization (3) and state extension (4), we derive an quadratic upper bound on the infinite horizon cost (8) of the form $z_k^T P_k^z z_k$. To this end, we calculate offline symmetric positive definite matrices $P_k^z = P_{k+N}^z$ such that

$$\begin{aligned} z_k^T P_k^z z_k &\geq J_{\infty,k}^z(z_k) \\ &= \sum_{i=0}^{\infty} z_{k+i|k}^T (\Gamma_x^T Q \Gamma_x + \Gamma_{u,k+i}^T R \Gamma_{u,k+i}) z_{k+i|k} \end{aligned} \quad (9)$$

in which

$$\Gamma_x = [I_n \quad 0], \quad \Gamma_{u,k+i} = [K_{k+i} \quad I_m \quad 0]. \quad (10)$$

Proposition 1 provides a way to calculate matrices P_i^z, $i = 0, 1, ..., N-1$, such that the upper bound condition (9) holds for any z_k and such that the trace of P_0^z is minimized and therefore giving a preferably small upper bound.

Proposition 1 *Choose $Q = Q^T > 0$ and $R = R^T > 0$ in order to weight the state and the input in objective function* (8). *The matrices $P_i^z = X_i^{z-1}$, $i = 0, 1, ..., N-1$, obtained by solving the convex optimization problem*

$$\min_{\alpha^P, X_i^z} \alpha^P, \qquad \textit{s.t.} \qquad \begin{bmatrix} \alpha^P I & * \\ I & X_0^z \end{bmatrix} \geq 0, \quad (11)$$

$$\begin{bmatrix} X_i^z & * & * & * \\ \Phi_i X_i^z & X_{i+1}^z & * & * \\ Q^{\frac{1}{2}}\Gamma_x X_i^z & 0 & I & * \\ R^{\frac{1}{2}}\Gamma_{u,i} X_i^z & 0 & 0 & I \end{bmatrix} \geq 0, \quad (12)$$

$$X_0^z = X_N^z, \quad (13)$$

satisfy (9) *for any z_k and the trace of P_0^z is minimized.*

Proof: Substituting $P_i^z = X_i^{z-1}$ and applying Schur complement to LMI condition (11) yields $\tilde{P}_0^z = \alpha^P I - P_0^z \geq 0$. Since $\tilde{P}_0^z$ is required to be positive semidefinite, its trace satisfies

$$\text{trace}(\tilde{P}_0^z) = \sum_{i=1}^{n+n_c m} \alpha^P - P_{0_{(i,i)}}^z = (n + n_c m)\alpha^P - \text{trace}(P_0^z) \geq 0.$$

Thus, minimizing α^P implies minimization of the trace of P_0^z.

Next, we prove $z_k^T P_0^z z_k$ to be an upper bound on the objective for any z_k as in (9). LMI (12) implies

$$P_i^z - \Gamma_x^T Q \Gamma_x - \Gamma_{u,i}^T R \Gamma_{u,i} - \Phi_i^T P_{i+1}^z \Phi_i \geq 0. \quad (14)$$

From (14) it directly follows that

$$\begin{aligned} & z_{k+i+1}^T P_{i+1}^z z_{k+i+1} - z_{k+i}^T P_i^z z_{k+i} \\ & \leq -z_{k+i}^T (\Gamma_x^T Q \Gamma_x + \Gamma_{u,i}^T R \Gamma_{u,i}) z_{k+i} \end{aligned} \quad (15)$$

for $i = 0, 1, ..., N-1$ and for any z_{k+i}. Summation of (15) from $i = 0$ to $i = \infty$ yields $z_k^T P_0^z z_k \geq J_{\infty,k}^z(z_k)$. ■

IV. Periodic Invariant Sets

The notion of periodic invariance plays a crucial role in our MPC scheme. Hence, the formal definition is given and techniques to calculate periodic invariant ellipsoidal and polyhedral sets are presented.

Definition 1 (Periodic Invariance) *Consider the extended autonomous periodic system* (5) *and the sets* S_i *satisfying* $S_{i+N} = S_i$. *If for* $i = 0, 1, ..., N-1$

$$\forall z \in S_i \;\Rightarrow\; \Phi_i z \in S_{i+1}\,, \tag{16}$$

then the sets S_i *are called periodic invariant.*

Figure 1 illustrates exemplary periodic invariant sets and corresponding trajectories for period $N = 3$. Note that trajectories might well leave each set S_i and only have to enter the sets after a complete time period.

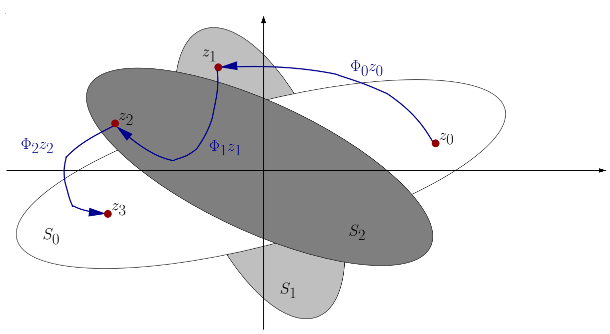

Fig. 1. Periodic Invariant Sets and Corresponding Trajectories.

In the following, two algorithms are given in order to calculate periodic invariant sets. Ellipsoidal invariant sets, which are also used in [4, 26], are considered in IV-A. Since the constraint sets are given in the form of polyhedral sets, polyhedral invariant sets might be significantly larger than ellipsoidal ones. Hence, the calculation of polyhedral invariant sets is explained in IV-B.

A. Ellipsoidal Periodic Invariant Sets

Algorithm 1 calculates periodic invariant ellipsoidal sets

$$S^z_{\epsilon,i} = \left\{ z \mid z^T X_i^{z-1} z \le 1 \right\}, \quad S^z_{\epsilon,i} = S^z_{\epsilon,i+N}, \tag{17}$$

which are subsets of the constraint set $S^z_{\epsilon,i} \subseteq S^z_{C,i}$, $i = 0, 1, ..., N-1$. Furthermore, the size of the set $S^{xz}_{\epsilon,0} = \left\{ x \mid x^T X_0^{xz-1} x \le 1 \right\}$, with $X_0^{xz-1} = T X_0^{z-1} T^T$, which is the projection of $S^z_{\epsilon,0}$ onto $\mathbb{R}^n$, is maximized in the sense of maximum trace of matrix X_0^{xz}.

Algorithm 1 (Ellipsoidal Periodic Invariant Sets)
Solve the semidefinite program

$$\min_{\alpha^X, X_i^z} \; \alpha^X, \qquad \text{s.t.} \quad \begin{bmatrix} \alpha^X I & * \\ T^T & X_0^z \end{bmatrix} \ge 0, \tag{18}$$

$$\begin{bmatrix} X_i^z & * \\ \Phi_i X_i^z & X_{i+1}^z \end{bmatrix} \ge 0, \tag{19}$$

$$\begin{bmatrix} 1 & * \\ (e_c \Psi^z_{C,i} X_i^z)^T & X_i^z \end{bmatrix} \ge 0, \tag{20}$$

$$X_N^z = X_0^z \tag{21}$$

$$i = 0, 1, \dots, N-1, \; c = 1, 2, \dots, c_x + c_u,$$

where $T = [I_n \; 0_{n \times n_c m}]$ *and* e_c *denotes the* c*-th row of the* $c_x + c_u$ *order identity. Return* $S^z_{\epsilon,i} = \left\{ z \mid z^T X_i^{z-1} z \le 1 \right\}$, $i = 0, 1, ..., N-1$.

Proof: Applying Schur complement to (18) yields $\tilde{X_0^{xz}} = X_0^{xz} - \frac{1}{\alpha^X} I \ge 0$. Since $\tilde{X}_0^{xz}$ is required to be positive semidefinite, its trace satifies

$$\text{trace}(\tilde{X}_0^{xz}) = \sum_{i=1}^{n} X^z_{0_{(i,i)}} - \frac{1}{\alpha^X} = \text{trace}(X_0^{xz}) - \frac{n}{\alpha^X} \ge 0.$$

Thus, minimizing α^X implies maximization of the trace of X_0^{xz}. For the proof of periodic invariance, LMI conditions (19) are rewritten for $i = 0, 1, ..., N-1$ as

$$z_{i+1}^T X_{i+1}^{z-1} z_{i+1} - z_i^T X_i^{z-1} z_i \le 0.$$

Hence,for $z_0 \in S^z_{\epsilon,0}$ and $X_i^{z-1} = X_{i+N}^{z-1}$

$$\begin{aligned} 1 \ge z_0^T X_0^{z-1} z_0 &> z_1^T X_1^{z-1} z_1 > \ldots > z_N^T X_N^{z-1} z_N \\ &= z_N^T X_0^{z-1} z_N > z_{N+1}^T X_1^{z-1} z_{N+1} > \ldots \end{aligned}$$

what shows the periodic invariance. LMI conditions (20) ensure $S^z_{\epsilon,i} \subseteq S^z_{C,i}$, $i = 0, 1, ..., N-1$, for details cf. [8]. In combination with the periodic invariance property, constraint satisfaction follows for all times. ∎

B. Polyhedral Periodic Invariant Sets

In this section, we present an algorithm to calculate offline polyhedral periodic invariant sets of maximum size satisfying the constraints. The algorithm is based on the results in [22] for polyhedral invariant sets for uncertain systems. Consider the extended autonomous periodic system (5) which is bounded by the polyhedral constraint sets (7). Algorithm 2 returns the periodic invariant sets

$$S^z_{\Pi,i} = \{ z \mid \Pi_i^z z \le 1 \}\,, \quad S^z_{\Pi,i} = S^z_{\Pi,i+N}, \tag{22}$$

satisfying $S^z_{\Pi,i} \subseteq S^z_{C,i}, \; i = 0, 1, ..., N-1$.

Algorithm 2 (Polyhedral Periodic Invariant Sets)
(i) *for* $j = 0, 1, ..., N-1$
  set $\Pi_j^z = \Psi^z_{C,j}$
(ii) *set* $c = 0$
  while $c < N$
    set $j = j+1$
    set $\Pi_j^z = \Pi_{j-N}^z$
    set $r =$ *number of rows in* Π^z_{j-N+1}
    set $p = 1$
    while $p \le r$

set e_p *as the* p*-th row of the* r*-th order identity*

$$\beta = \max_z \; e_p \Pi^z_{j-N+1} \Phi_j z, \;\; s.t. \; \Pi^z_j z \leq 1 \tag{23}$$

if $\beta > 1$ *or problem* (23) *is unbounded*

set $\Pi^z_j = \begin{bmatrix} \Pi^z_j \\ e_p \Pi^z_{j-N+1} \Phi_j \end{bmatrix}$

set $p = p + 1$

if $\Pi^z_j = \Pi^z_{j-N}$

set $c = c + 1$

else

set $c = 0$

(iii) *rename and sort* Π^z_{j-i}*,* $i = 0, 1, ..., N-1$*, by applying modulo* N *to the index*

return $S^z_{\Pi,i} = \{z | \Pi^z_i z \leq 1\}$*,* $i = 0, 1, ..., N-1$.

Proof: In the first step of the algorithm, the polyhedral sets are set to the constraint set $S^z_{\Pi,i} = S^z_{C,i}$, $i = 0, 1, ..., N-1$. In the following steps, only new constraints are added but none are removed. This directly proves $S^z_{\Pi,i} \subseteq S^z_{C,i}$. The algorithm's inner 'while $p \leq r$'-loop adds new constraints to the set $S^z_{\Pi,j-N}$ in the j-th iteration to generate set $S^z_{\Pi,j}$ such that

$$\forall z \in S^z_{\Pi,j} \Rightarrow \Phi_j z \in S^z_{\Pi,j-N+1}. \tag{24}$$

Note the similarity between (24) and (16). An abortion criterion to stop the algorithm in the case that the periodic invariant sets are found is deposited in the outer while-loop. Accordingly, the algorithm has to be stopped when

$$S^z_{\Pi,j+i} = S^z_{\Pi,j-N+i} \quad \text{for} \quad i = 0, 1, ..., N-1 \tag{25}$$

is found. Periodic invariance in the sense of Definition 1 is directly proven for the achieved sets when substituting $S^z_{\Pi,j-N+1}$ by $S^z_{\Pi,j+1}$ in the argument of (24). ■

V. MPC USING FIXED DETERMINED INVARIANT SETS

In this section, we present the outline for the proposed MPC scheme for periodic systems. The scheme is divided into an offline and an online part. In the offline part, a periodic linear feedback law is determined first. Then, using the control parametrization (3) and the extended state (4), a quadratic upper bound on the infinite horizon cost (8) and periodic invariant sets in extended state coordinates are calculated. In the online part, at each recalculation instant, the quadratic upper bound on the infinite horizon cost (8) is minimized. For this purpose, only the free superimposed control moves are optimization variables and have to be calculated such that the extended state is contained in the first set of the offline calculated periodic invariant sets. In the following, the offline part and two variants of the online part are presented. The first variant uses ellipsoidal invariant sets and the second one uses polyhedral invariant sets.

Algorithm 3 (Periodic MPC using fixed determined sets)
Offline Part. *Calculate for periodic system* (1) *a linear periodic feedback* $u_k = K_k x_k$ *with* $K_k = K_{k+N}$ *which stabilizes the system in absence of constraints.*

Choose the number of superimposed control moves $\underline{c}_{k+\cdot|k} = [c^T_{k|k} \; c^T_{k+1|k} \; ... \; c^T_{k+n_c-1|k}]^T$ *as a multiple of the period, i.e.* $n_c = n_p N$ *with* $n_p \in \mathbb{N}_+$.

Consider control parametrization (3) *and state extension* (4) *and derive the extended autonomous periodic system* (5) *which is bounded by the polyhedral constraint sets* (7).

For objective function (8) *with* $Q = Q^T > 0$ *and* $R = R^T > 0$*, apply Proposition 1 to calculate* P^z_0 *satisfying the upper bound condition* (9).

Application of Algorithm 1 or Algorithm 2 returns periodic invariant sets which are contained within the imposed constraint sets of (7).

A. Online Part for Ellipsoidal Invariant Sets. *Take* P^z_0 *and* X^z_0 *obtained in the offline part. Define* $T_{11} = [I_n \quad 0_{n \times n_c m}]$ *and* $T_{22} = [0_{n \times n} \quad I_{n \times n_c m}]$ *in order to select the blocks*

$$\begin{aligned} {X^z_{0,11}}^{-1} &= T_{11} {X^z_0}^{-1} T^T_{11}, \\ {X^z_{0,12}}^{-1} &= T_{22} {X^z_0}^{-1} T^T_{11}, \\ {X^z_{0,22}}^{-1} &= T_{22} {X^z_0}^{-1} T^T_{22}. \end{aligned} \tag{26}$$

Solve at discrete times $k = 0, N, ..., rN$*,* $r \in \mathbb{N}_+$*, and for measured state* $x_k = x_{k|k}$ *the semidefinite optimization problem*

$$\min_{\alpha^P_k, \underline{c}_{k+\cdot|k}} \alpha^P_k, \qquad s.t. \qquad \begin{bmatrix} \alpha^P_k & * \\ [x^T_k \; \underline{c}^T_{k+\cdot|k}]^T & P^z_0 \end{bmatrix} \geq 0, \tag{27}$$

$$\begin{bmatrix} 1 - x^T_k {X^z_{0,11}}^{-1} x_k - 2 x^T_k {X^z_{0,12}}^{-1} \underline{c}_{k+\cdot|k} & * \\ \underline{c}_{k+\cdot|k} & X^z_{0,22} \end{bmatrix} \geq 0. \tag{28}$$

Until the next recalculation instant, apply the control law

$$u_{k+i|k} \quad = \quad K_{k+i} x_{k+i} + c_{k+i|k} \tag{29}$$

to the system.

B. Online Part for Polyhedral Invariant Sets. *Take* P^z_0 *and* Π^z_0 *obtained in the offline part. Solve at discrete times* $k = 0, N, ..., rN$*,* $r \in \mathbb{N}_+$*, and for measured state* $x_k = x_{k|k}$ *the quadratic optimization problem*

$$\min_{\underline{c}_{k+\cdot|k}} \quad [x^T_k \quad \underline{c}^T_{k+\cdot|k}] P^z_0 [x^T_k \quad \underline{c}^T_{k+\cdot|k}]^T \qquad s.t. \tag{30}$$

$$\Pi^z_0 \begin{bmatrix} x_k \\ \underline{c}_{k+\cdot|k} \end{bmatrix} \leq 1. \tag{31}$$

Until the next recalculation instant, apply the control law

$$u_{k+i|k} \quad = \quad K_{k+i} x_{k+i} + c_{k+i|k} \tag{32}$$

to the system.

Theorem 2 *Assume the online part of Algorithm 3 is feasible at the initial time* $k = 0$*. The model predictive controller given by* (29) *or* (32)*, respectively, in Algorithm 3 for the linear periodic system* (1) *has the following properties.* (i) *The free control moves* $c_{k+i|k}$ *minimize the upper bound of objective function* (9) *at time instant* k *for initial condition* x_k*,* (ii) *the corresponding optimization in the online part of Algorithm 3 is feasible at all future recalculation times* $k \geq 0$*,* (iii) *the closed loop system is asymptotically stable,* (iv) *the constraints* (2) *are satisfied for all times* $k \geq 0$.

Proof: (i.A.) Applying Schur complement to LMI condition (27) in combination with Proposition 1 gives

$$\alpha^P_k \geq [x^T_k \quad \underline{c}^T_{k+\cdot|k}] P^z_0 [x^T_k \quad \underline{c}^T_{k+\cdot|k}]^T \geq J^z_{\infty,k}. \tag{33}$$

Hence, minimizing α_k^P implies minimization of the upper bound $z_k^T P_0^z z_k$ on cost function $J_{\infty,k}^z$.

(i.B.) The proof follows directly from Proposition 1.

(ii) Provided feasibility at recalculation instant k, the online algorithms return the superimposed control moves

$$\underline{c}_{k+\cdot|k} = [c_{k|k}^T \; c_{k+1|k}^T \; \dots \; c_{k+n_c-1|k}^T]^T. \tag{34}$$

Further, we assume no disturbances and model plant mismatches. Therefore, $x_{k+N} = x_{k+N|k}$ when the control input calculated in Algorithm 3 is applied at discrete times $k+i$ for $i = 0, 1, ..., N-1$. Then, the extended state prediction at discrete time $k+N$ reads

$$z_{k+N} = [x_{k+N}^T \quad \underline{c}_{k+N+\cdot|k}^T]^T \tag{35}$$

in which

$$\underline{c}_{k+N+\cdot|k} = [c_{k+N|k}^T \; c_{k+N+1|k}^T \; \dots \; c_{k+n_c-1|k}^T \; 0_{Nm\times 1}^T]^T. \tag{36}$$

It is now shown that $\underline{c}_{k+N+\cdot|k}$ is a feasible solution to the optimization problem at time $k+N$. At discrete time $k+N$ the online algorithms constraints (28) and (31) demand $z_{k+N} \in S_{\epsilon,0}^z$ and $z_{k+N} \in S_{\Pi,0}^z$, respectively. This is directly satisfied because of the periodic invariance property. It remains to show satisfaction of (27) at time $k+N$, which is equivalent to $\alpha_{k+N}^P \geq [x_{k+N}^T \quad \underline{c}_{k+N+\cdot|k}^T] P_0^z [x_{k+N}^T \quad \underline{c}_{k+N+\cdot|k}^T]^T$. Since the scalar α_{k+N}^P is a non-bounded problem variable, it is always possible to find such α_{k+N}^P. Using induction feasibility follows for all future times. This completes the proof of (ii).

(iii) At recalculation instant k and for measured state $x_k = x_{k|k}$, the online algorithm returns the optimal superimposed control moves $\underline{c}_{k+\cdot|k}^o$. Applying the online algorithm at next recalculation instant, i.e. at $k+N$, returns the optimal superimposed control moves $\underline{c}_{k+N\cdot|k+N}^o$.

As was shown in (ii), the control moves

$$\underline{c}_{k+N+\cdot|k}^o = [c_{k+N|k}^{oT} \; c_{k+N+1|k}^{oT} \dots c_{k+N+n_c-1|k}^{oT} \; 0_{Nm\times 1}^T] \tag{37}$$

obtained at recalculation instant k are a feasible solution at recalculation instant $k+N$. However, this solution is in general non-optimal. Assuming no model plant mismatch, i.e. $x_{k+N} = x_{k+N|k+N} = x_{k+N|k}$,

$$z_{k+N|k} = [x_{k+N}^T \quad \underline{c}_{k+N+\cdot|k}^{oT}]^T$$

when using the non-optimal solution (37) whereas

$$z_{k+N|k+N} = [x_{k+N}^T \quad \underline{c}_{k+N+\cdot|k+N}^{oT}]^T$$

when utilizing the optimal solution. Consequently, as $z_{k+N|k}$ is feasible, but not necessarily optimal,

$$z_{k+N|k}^T P_0^z z_{k+N|k} \geq z_{k+N|k+N}^T P_0^z z_{k+N|k+N}. \tag{38}$$

Proposition 1 yields for any $i \geq 0$

$$\begin{aligned} & z_{k+i+1|k}^T P_{k+i+1}^z z_{k+i+1|k} - z_{k+i|k}^T P_{k+i}^z z_{k+i|k} \\ & \qquad \leq -z_{k+i|k}^T (\Gamma_x^T Q \Gamma_x + \Gamma_{u,k+i}^T R \Gamma_{u,k+i}) z_{k+i|k}. \end{aligned}$$

Due to the positive definiteness of Q and R and using (38), the following sequence of inequalities holds

$$\begin{aligned} & z_{k|k}^T P_k^z z_{k|k} > z_{k+1|k}^T P_{k+1}^z z_{k+1|k} > \dots \\ & \quad > z_{k+N|k}^T P_{k+N}^z z_{k+N|k} \geq z_{k+N|k+N}^T P_k^z z_{k+N|k+N} \\ & \quad > z_{k+N+1|k+N}^T P_{k+1}^z z_{k+N+1|k+N} > \dots. \end{aligned}$$

Thus, the sequence $z_k^T P_k^z z_k$, $P_k^z = P_{k+N}^z$, is strictly decreasing along the trajectories of the extended system and $V(z_k) = z_k^T P_k^z z_k$ is a periodic Lyapunov function of the extended system. This directly shows asymptotic stability via the periodic Lyapunov lemma [2, 6].

(iv) Constraint satisfaction follows immediately from the requirement of $z_k \in S_{\epsilon,0}^z$ and $z_k \in S_{\Pi,0}^z$, respectively, in combination with the fact that the periodic invariant sets calculated offline are contained within the imposed constraint sets. ■

VI. Simulation Results

In this section, the benefits of the presented MPC scheme for periodic systems are illustrated. In Subsection VI-A, the MPC scheme is compared to the existing scheme in [4] regarding computation time and performance. In Subsection VI-B, the ellipsoidal set and the polyhedral set approach are compared with respect to the sizes of the resulting invariant sets.

A. Computation Time and Performance

For the simulations in this subsection, we use the constrained periodic system from [4]. The pre-determined feedback $u_k = K_k x_k$ is chosen as

$$K_1 = \begin{bmatrix} -0.133 & -0.221 & -0.073 \\ -0.268 & -0.454 & -0.143 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -0.067 & -0.180 & -0.137 \\ -0.175 & -0.468 & -0.363 \end{bmatrix},$$

$$K_3 = \begin{bmatrix} -0.302 & -0.276 & -0.186 \\ -0.134 & -0.114 & -0.089 \end{bmatrix}.$$

The weights in the infinite horizon cost (8) are chosen as $Q = 0.1 \cdot I_3$ and $R = 5 \cdot I_2$. The initial condition is given by $\bar{x}_0 = [10 \; 10 \; 0]^T$. Since $\bar{x}_0$ has to be contained within the appropriate projection S_0^{xz} of the invariant sets, the number of superimposed control moves has to be chosen as $n_c = 9$ for the ellipsoidal sets and $n_c = 3$ for the polyhedral sets. In Figure 2, the results for performance and computation time for the presented MPC scheme using ellipsoidal and polyhedral invariant sets and the existing scheme by Böhm et al. [4] are shown. Note performance is evaluated using the objective function $J = \sum_{k=0}^{15} x_k^T Q x_k + u_k^T R u_k$.

The new MPC scheme using polyhedral sets clearly shows the best results. It achieves best performance and a significantly reduced computation time compared to the existing scheme in [4]. The scheme using ellipsoidal sets achieves a slightly worse performance but still reduced computation time compared to [4]. These demonstrate the benefits of the pre-computed control law in the MPC scheme.

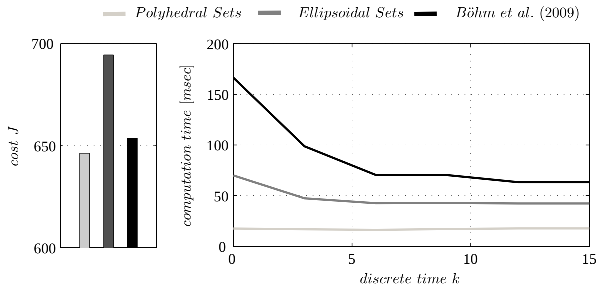

Fig. 2. Performance and Computation Times.

B. Sizes of Invariant Sets

For a better visualization of the invariant sets, we consider a second order system defined by the matrices

$$A_0 = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix},$$

which is input constrained by $-1 \leq u \leq 0.5$. The predetermined stabilizing feedback is chosen as $u_k = K_k x_k$ with $K_1 = [-0.646 \quad -0704]$, $K_2 = [-0.405 \quad -0.62]$. The sizes of the invariant sets are investigated for different numbers n_c of superimposed control moves by application of Algorithm1 and Algorithm 2, respectively. Figure 3 illustrates the projections S_0^{xz} of the ellipsoidal and polyhedral sets onto $\mathbb{R}^n$. As can be expected, the polyhedral sets are

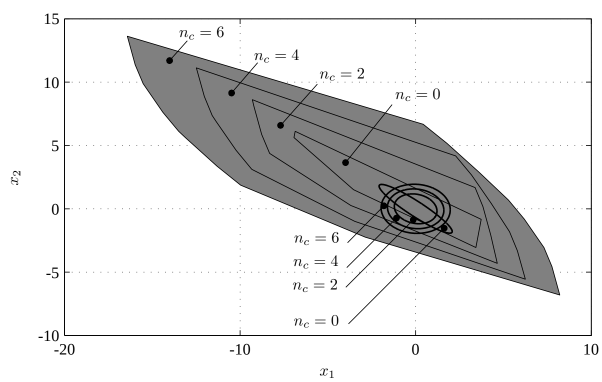

Fig. 3. Invariant Sets for Different Number of Control Moves.

significantly larger than the corresponding ellipsoidal sets. It is especially noticeable that the ellipsoidal sets are not able to cope with the non-symmetric constraints. Furthermore, a considerable enlargement of the sets by inreasing the number of available control moves can be observed.

VII. Conclusions

In this paper a novel model predictive control scheme for the stabilization of constrained linear dicrete-time periodic systems is presented. Similar to the work in [19], a linear periodically time-varying feedback law is calculated offline. Superimposed control moves are calculated online as a solution of a convex optimization problem in order to achieve constraint satisfaction and improve performance. The superimposed control moves are required to be contained in a periodic invariant set, which is also calculated offline. Two types of invariant sets have been investigated, ellipsoidal and polyhedral sets. A numerical example shows the effectiveness of the proposed scheme and a significant reduction in computational demand compared to the existing MPC scheme in [4]. Furthermore, the advantages of the polyhedral set approach compared to the use of ellipsoidal sets has been demonstrated.

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