Purity Filtration and the Fine Structure of Autonomy

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This paper is dedicated to Professor ULRICH OBERST
on the occasion of his 70th birthday.

***Abstract*— This paper announces a constructive setup for homological algebra (of categories of finitely presented modules) in which the CARTAN-EILENBERG resolution of complexes and a particular GROTHENDIECK spectral sequence can be used to compute the purity filtration of a module M (associated to a system Σ). The purity filtration yields the fine structure of the torsion submodule of M, which corresponds to the autonomous part of the system Σ obstructing its controlability.**

I. INTRODUCTION

The idea of viewing a linear system of PDEs as a module over an appropriate ring of differential operators was emphasized by B. MALGRANGE[1] in the late 1960's. But it wasn't until the early 90's until it became clear (see for example [Obe90], [Fli90], [Mou95], [Zer00], to name a few) how a linear control system Σ (without boundary conditions) can be studied in terms of an **associated** module M_Σ over some suitable ring D. This insight allowed an extensive use of homological algebra (see for example [Qua99], [PQ99], [CQR05], [QR08]) to characterize and clarify various system theoretic properties.

In case D is an ORE domain it is by now well-known that the **autonomous part** of a system Σ corresponds to the **torsion submodule** $\operatorname{tor} M$ of the associated system module $M := M_\Sigma$, where $\operatorname{tor} M$ is classically defined as

$$\operatorname{tor} M := \{m \in M \mid \exists d \in D \setminus \{0\} : dm = 0\}.$$

If D is commutative then dropping "domain" means replacing "$\exists d \in D \setminus \{0\}$" by "$\exists d$ not a zero divisor".

For a *finitely generated* D-module M the torsion submodule coincides with the kernel of the **evaluation map**

$$\begin{array}{rcl} \varepsilon : M & \to & M^{**} := \operatorname{Hom}_D(\operatorname{Hom}_D(M,D),D) \\ m & \mapsto & (\varphi \mapsto \varphi(m)). \end{array}$$

Taking

$$\operatorname{tor} M = \ker \varepsilon,$$

as the definition of the torsion submodule of the *finitely generated*[2] D-module M avoids imposing further restrictions on the ring: D is an associative (*not* necessarily commutative) ring with 1. **From now on all modules will be assumed finitely generated over** D.

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[1]... and according to him goes back to EMMY NOETHER.

[2]Recall, $\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Z}) = 0$ for M divisible, even if M is torsion-free, e.g. $M = \mathbb{Q}$. Modules with injective evaluation map are called **torsionless**. So $\mathbb{Q}$ is a torsion-free $\mathbb{Z}$-module, which is *not* torsionless. Both notions coincide for finitely generated modules over ORE domains.

The torsion submodule gives rise to the short exact sequence

$$0 \to \operatorname{tor} M \to M \to M/\operatorname{tor} M \to 0$$

describing the (system) module M as an **extension** of the **torsion-free factor** (or **controllable part**) $M/\operatorname{tor} M$ by the torsion submodule (or autonomous part) $\operatorname{tor} M$. This extension defines a **2-step filtration** $0 \leq \operatorname{tor} M \leq M$ of the module M.

In the case when D is a commutative NOETHERian ring one can see in an elementary, geometrically motivated way, how this 2-filtration can be refined into a **$(1 + \dim D)$-step filtration**, called the **purity filtration**, where $\dim D$ is the KRULL dimension[3] of D. More precisely, the torsion submodule further admits a $(\dim D)$-step filtration.

Recall that the **KRULL dimension** $\dim D$ of a commutative ring D with 1 is defined to be the supremum of the heights of all prime ideals of D, where the **height** $\operatorname{ht}\mathfrak{p}$ of a prime ideal $\mathfrak{p}$ is the supremum of all integers d such that there exists chain of prime ideals $\mathfrak{p} > \mathfrak{p}_0 > \cdots > \mathfrak{p}_d$ [Har77, Def. on p. 6]. For example, the KRULL dimension of a field k is zero, $\dim \mathbb{Z} = 1$, and $\dim R[x_1, \ldots, x_n] = \dim R + n$ for R NOETHERian.

The definition of the KRULL dimension is then extended to nontrivial D-modules using

$$\dim M := \dim \frac{D}{\operatorname{Ann}_D(M)}.$$

Define the **codimension** of a nontrivial module M as

$$\begin{array}{rcl} \operatorname{codim} M & := & \min\{\operatorname{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp} M\} \\ & = & \min\{\operatorname{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Ass} M\}. \end{array}$$

and set the codimension of the zero module to be ∞. For $\dim D < \infty$ the definition simplifies to

$$\operatorname{codim} M := \dim D - \dim M.$$

Remark 1.1: Let D be a commutative NOETHERIAN domain. Then M is torsion (i.e., $M = \operatorname{tor} M$) $\iff$ $\operatorname{Ann}_D(M) \neq 0 \iff \operatorname{codim} M \geq 1$.

Definition 1.2 (Geometric definition): Let D be a commutative NOETHERian ring with 1 and M a D-module. Define the submodule $\operatorname{tor}_{-c} M$ as the biggest submodule of M of codimension $\geq c$. The *ascending* filtration

$$\cdots \leq \operatorname{tor}_{-(c+1)} M \leq \operatorname{tor}_{-c} M \leq \cdots \leq \operatorname{tor}_0 M := M$$

[3]NAGATA constructed a NOETHERian ring with infinite KRULL dimension.

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is called the **purity filtration** of M [HL97, Def. 1.1.4]. The graded part

$$M_c := \operatorname{tor}_{-c} / \operatorname{tor}_{-(c+1)}$$

is **pure** of codimension c, i.e. any nontrivial submodule of M_c has codimension c. $\operatorname{tor}_{-1} M$ is nothing but the torsion submodule $\operatorname{tor} M$. This suggests calling $\operatorname{tor}_{-c} M$ the **c-th (higher) torsion submodule** of M.

For a fixed codimension $c \geq 1$ the full subcategory of modules of codimension $\geq c$ is a **localizing category** in the sense of GABRIEL (cf. [Obe10]).

II. HISTORY OF THE PURITY FILTRATION

It is known since the pioneering work of J.-E. ROOS [Roo62] that the **purity filtration** of the module M can be computed using the **bidualizing spectral sequence**:

$$E^2_{pq} = \operatorname{Ext}^{-p}(\operatorname{Ext}^q(M, D), D) \Rightarrow \begin{cases} M & \text{for } p+q=0, \\ 0 & \text{otherwise.} \end{cases}$$

For the bidualizing spectral sequence to be bounded one only needs to assume that either the left or right global dimension[4] of D is finite. The most powerful aspect of this construction is that it works without the commutativity assumption on the ring D.

The bidualizing spectral sequence is a special case of a **GROTHENDIECK spectral sequence**[5]

$$E^2_{pq} = (\mathrm{R}^{-p}F \circ \mathrm{R}^q G)(M) \Longrightarrow \mathrm{L}_{p+q}(F \circ G)(M)$$

of two composable contravariant functors F and G applied to a left D-module M: Take $F := \operatorname{Hom}_D(-, D_D)$ and $G := \operatorname{Hom}_D(-, {}_D D)$. It was introduced five years earlier in GROTHENDIECK's seminal Tôhuku paper [Gro57].

Other early references to the purity filtration are M. KASHIWARA's master thesis (December 1970) [Kas95, Theorem 3.2.5] on algebraic D-modules and J.-E. Björk's standard reference [Bjö79, Chap. 2, Thm. 4.15]. All these references address the construction of this filtration from a homological[6] point of view, where the assumption of commutativity of the ring D can be dropped.

Definition 2.1 (Homological definition): Let D be a ring (associative with 1) with finite left or right global dimension and M a finitely generated D-module. Then, the purity filtration on M is the one induced by the bidualizing spectral sequence.

I am not aware of a "geometric" definition in this generality.

III. HOW TO COMPUTE THE PURITY FILTRATION?

A. Spectral sequences

Definition 2.1 characterizes the purity filtration using the bidualizing spectral sequence. This homological description would be useless from the constructive point of view, if one were not able to effectively compute spectral sequences. The latter are usually introduced in the context of module categories, where diagram chasing of elements is used to establish their existence. But it is known that spectral sequences exist even in the more general context of ABELian categories, where diagram chasing of elements cannot be used, since objects of such categories are not necessarily sets (at least a priori; see the discussion in [Har77, §III.1, p. 203]). The algorithmic treatment of spectral sequences in [Bar] was therefore based on operations on morphisms in an ABELian category rather than chasing elements through diagrams. The key notion introduced in [Bar] is that of a **generalized morphism**. It used to turn various algorithms into closed formulas and to algorithmically solve the extension problem of spectral sequences at abutment. Finally, in [BLH] it shown that the computability of the ABELian category of finitely presented modules over a ring follows from the computability of the ring, where a ring is called **computable** if one can algorithmically solve one-sided (in)homogenous linear systems with entries in the ring.

B. CARTAN-EILENBERG *resolution*

In fact, the purity filtration can indeed be computed without performing a complete spectral sequence algorithm. The aim of this subsection is to clarify this point, while the next section indicates why a complete spectral sequence algorithm is still indispensable for extracting more information about the graded parts of the purity filtration and hence about the module M itself. Moreover, spectral sequences were invented to process large (co)homology computations by chopping them into several smaller pieces.

The first step of computing the bidualizing spectral sequence of a module M is to compute the GROTHENDIECK bicomplex (see below) for the two composable functors $F := \operatorname{Hom}_D(-, D_D)$ and $G := \operatorname{Hom}_D(-, {}_D D)$ using the so-called CARTAN-EILENBERG resolution. This goes as follows:

1) Compute a projective (or free) resolution $M_\bullet := P^M_\bullet$ of M.
2) Apply the contravariant (inner) functor G to $M_\bullet$ and obtain the cocomplex $G(M_\bullet)$.
3) Compute the **CARTAN-EILENBERG resolution** of the cocomplex $G(M_\bullet)$ [Wei94, §5.7], [Bar, §7]. Using the sign-trick this resolution (which is a cocomplex of complexes) can be viewed as fourth quadrant cohomological bicomplex $CE^{\bullet\bullet}$, called the **CARTAN-EILENBERG bicomplex**.

[4]Recall, the **left global (homological) dimension** is the supremum over all projective dimensions of *left* D-modules. If D is left NOETHERian, then the left global dimension of D coincides with the **weak global (homological) dimension**, which is the largest integer μ such that $\operatorname{Tor}^D_\mu(M, N) \neq 0$ for some right module M and left module N, otherwise infinity (cf. [MR01, 7.1.9]). This last definition is obviously left-right symmetric. The same is valid if "left" is replaced by "right".

[5]Although a special **hyper-derived functor spectral sequence**

$${}^{\mathrm{II}}E^2_{pq} = \mathrm{R}^{-p}F(H^q(M^\bullet)) \Rightarrow \mathbb{R}^{p+q}F(M^\bullet) \Leftarrow H_p(\mathrm{R}^{-q}F(M^\bullet)) = {}^{\mathrm{I}}E^2_{pq}$$

(take $M^\bullet := G(P^M_\bullet)$, where $P^M_\bullet$ is a projective resolution of M), it turned out that most hyper-derived functor spectral sequences discussed in [CE99] are indeed GROTHENDIECK spectral sequences.

[6]KASHIWARA did not use spectral sequences: "Instead of using spectral sequences, Sato devised [...] a method using associated cohomology", [Kas95, Section 3.2].

4) Finally, apply the contravariant (outer) functor F to the bicomplex $CE^{\bullet\bullet}$ and obtain the fourth quadrant homological bicomplex $B_{\bullet\bullet}$, called the **GROTHENDIECK bicomplex** of M (or $M_\bullet$), F, and G.

Denote by $T_\bullet := \mathrm{Tot}(B_{\bullet\bullet})$ the **total complex** of the bicomplex $B_{\bullet\bullet}$ with objects $\mathrm{Tot}_n(B_{\bullet\bullet}) := \bigoplus_{p+q=n} B_{pq}$. $T_\bullet$ has remarkable properties:

a) $T_\bullet$ is exact except in degree 0, where its homology is naturally isomorphic to M. This follows easily from the collapsing of the (first) spectral sequence associated to the GROTHENDIECK bicomplex $B_{\bullet\bullet}$ (cf. [Wei94, §5.8], [Bar, §8]).

b) $T_\bullet$ is the total complex of $B_{\bullet\bullet}$ but also of the **transposed** bicomplex ${}^{\mathrm{t}}B_{\bullet\bullet}$ (with objects ${}^{\mathrm{t}}B_{pq} := B_{qp}$), and is thus filtered in two different ways. It is elementary to see that a filtered complex induces filtrations on all its homology objects [Bar09]. While the first filtration of $T_\bullet$ induces the trivial filtration on M (being the 0-th homology of $T_\bullet$), the second filtration induces the desired purity filtration on M.

Steps 1)-4) and the construction of the total complex describe the analog of passing from M to its D-double dual $M^{**} = F(G(M))$ in the **derived category**. Although the module-theoretic evaluation map is generally neither surjective nor injective, its analog in the derived category is due to the first mentioned property of $T_\bullet$ an *isomorphism*.

Summing up, the purity filtration on $M \cong H_0(T_\bullet)$ is induced by the second filtration on $T_\bullet$ and can be computed as such, without a complete[7] spectral sequence algorithm.

C. Primary decomposition

It is also obvious that a primary decomposition algorithm of ideals over a polynomial ring $D = k[x_1, \ldots, x_n]$ would suffice to compute the purity filtration of a (cyclic) module M over D. Conversely, the (double) Ext modules of M with values in D, which appear in the second page of the bidualizing spectral sequence, can be used to compute an **equidimensional decomposition** of the support of M. This was indeed utilized in [EHV92] as the first step of a primary decomposition algorithm.

D. QUADRAT*'s recent approach*

QUADRAT recently introduced another constructive approach to the purity filtration [Qua10a], [Qua10b]. His approach is simpler in the sense that it does not make use of spectral sequences.

IV. WHY SPECTRAL SEQUENCES?

Spectral sequences have the reputation of being extremely complex and difficult to comprehend. The relatively big amount of data entering their definition is probably one of the reasons for this reputation. Below is an attempt to summarize some of the advantages of spectral sequences relevant to our particular context:

[7]Computing $B_{\bullet\bullet}$ is of course part of computing the bidualizing spectral sequence.

- Spectral sequences were invented to compute the (co)homology objects of a filtered complex $T_\bullet$ by an approximation process consisting of several *smaller* steps. These steps successively take deeper inter-level interaction between the graded parts of the complex into account [Bar09, §3]. Spectral sequences thus offer a computational advantage when dealing with large examples.
- Spectral sequences typically become intrinsic from a certain page on. The objects and morphisms appearing in all intrinsic pages of the spectral sequence can serve as invariants of the original data. In our context this means that all modules and maps in the pages E^a, for $a \geq 2$, of the bidualizing spectral sequence are invariants of the original module M (or $M_\bullet$). Numerical invariants can now be easily extracted. [Bar, §9.1.5] introduces such a numerical invariant called the **codegree of purity**. It is in this sense that spectral sequences can be seen as a goal rather than merely a means.
- Spectral sequences lead to very simple proofs of various results. See [Bar, §9.1.4] for some results related to the purity of a module.
- Spectral sequences are one of the unifying principles in homological algebra. There range of application goes far beyond a specific application [Rot09, §10]. In particular, they offer a unified way to describe a lot of important filtrations in algebra and geometry. A general implementation is thus highly desirable. See [Bar, §9.2] for another algorithmic application.

ROTMAN ends his book [Rot09] with the words:
"The reader should now be convinced that using spectral sequences can prove interesting theorems. Moreover, even if there are "elementary" proofs of these results (i.e., avoiding spectral sequences), these more "sophisticated" proofs offer a systematic approach in place of sporadic success."

V. EXAMPLES

The examples listed below are not of any physical significance. They are chosen to accumulate many algorithmic difficulties.

Example 5.1: Let $D = \mathbb{Q}[x, y, z]$ and $M := \operatorname{coker} A = D^{1\times 5}/D^{1\times 6}A$ for $A \in D^{6\times 5}$,

$$A = \begin{pmatrix} xy & yz & z & 0 & 0 \\ x^3z & x^2z^2 & 0 & xz^2 & -z^2 \\ x^4 & x^3z & 0 & x^2z & -xz \\ 0 & 0 & xy & -y^2 & x^2-1 \\ 0 & 0 & x^2z & -xyz & yz \\ 0 & 0 & x^2y-x^2 & -xy^2+xy & y^2-y \end{pmatrix}.$$

The triangulation algorithm `IsomorphismOfFiltration`, described in [Bar, Appendix A], applied to the 2-step filtration $0 \leq \operatorname{tor} M \leq M$ returns an isomorphism $\alpha_B : \operatorname{coker} B \to \operatorname{coker} A$ with an equivalent block triangular matrix $B = \left(\begin{array}{c|c} B_0 & B_{01} \\ \hline 0 & B_1 \end{array}\right) =$

$$\left(\begin{array}{cccc|ccc} 0 & 0 & x & -y & 0 & -1 & 0 \\ xy & yz & z & 0 & 0 & 0 & 0 \\ x^2 & xz & 0 & z & 0 & 0 & -1 \\ \hline \cdot & \cdot & \cdot & \cdot & 0 & z & y \\ \cdot & \cdot & \cdot & \cdot & yz & xz & 0 \\ \cdot & \cdot & \cdot & \cdot & -z^2 & 0 & xz \\ \cdot & \cdot & \cdot & \cdot & y^2-y & xy-x & 0 \\ \cdot & \cdot & \cdot & \cdot & x^2-1 & y & 0 \\ \cdot & \cdot & \cdot & \cdot & -xz & 0 & x^2 \end{array}\right) \in D^{9\times 7}$$

(i.e., $M \cong \operatorname{coker} B$ with $M_0 = M/\operatorname{tor} M \cong \operatorname{coker} B_0$ and $\operatorname{tor} M \cong \operatorname{coker} B_1$). The isomorphism α_B is represented by a matrix

$$V_B := \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -x & y & 0 \\ -x^2 & -xz & 0 & -z & 0 \end{pmatrix} \in D^{7\times 5}.$$

There exists a matrix U_B satisfying $A = U_B B V_B$. Finding the “coordinate change” matrix V_B is the involved part of the computation, whereas the matrix U_B can always be computed a posteriori.

The triangulation algorithm `IsomorphismOfFiltration` applied to the 4-step *purity* filtration now yields an isomorphism $\alpha_C : \operatorname{coker} C \to \operatorname{coker} A$ with an equivalent 12×9 block triangular matrix $C =$

$$\left(\begin{array}{cccc|ccc|c|c} 0 & 0 & x & -y & 0 & 1 & 0 & 0 & 0 \\ xy & -yz & -z & 0 & 0 & 0 & 0 & 0 & 0 \\ x^2 & -xz & 0 & -z & 1 & 0 & 0 & 0 & 0 \\ \hline \cdot & \cdot & \cdot & \cdot & y & -z & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & x & 0 & -z & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & 0 & x & -y & -1 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & -y & x^2-1 & 0 & 0 \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & z & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & y-1 & 0 \\ \hline \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & z \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & y \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & x \end{array}\right).$$

This means that $M \cong \operatorname{coker} C$ with $M_0 \cong \operatorname{coker} B_0$, $M_1 \cong \operatorname{coker} \begin{pmatrix} y & -z & 0 \\ x & 0 & -z \\ 0 & x & -y \\ 0 & -y & x^2-1 \end{pmatrix}$, $M_2 \cong \operatorname{coker} \begin{pmatrix} z \\ y-1 \end{pmatrix}$, and $M_3 \cong \operatorname{coker} \begin{pmatrix} z \\ y \\ x \end{pmatrix}$, for the higher torsion modules. The 1-pure subfactor module M_1 is supported on the surface $\operatorname{V}(\operatorname{Ann}_D(M_1)) = \operatorname{V}(y^2 - (x^3 - x))$, ruled over an elliptic curve. The isomorphism α_C is represented by a matrix

$$V_C := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ -x^2 & -xz & 0 & -z & 0 \\ 0 & 0 & x & -y & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & x^2 & -xy & y \\ x^3 & x^2 z & 0 & xz & -z \end{pmatrix} \in D^{9\times 5}.$$

There exists a matrix U_C satisfying $A = U_C C V_C$, which can now be easily computed.

Example 5.2: Let $D = \mathbb{Q}[x,y,z]\langle \partial_x, \partial_y, \partial_z \rangle$ denote the 3-dimensional Weyl algebra. Consider the finitely presented D-module $M := \operatorname{coker} A = D^{1\times 2}/D^{1\times 8}A$ for $A \in D^{8\times 2}$, $A =$

$$\begin{pmatrix} \partial_y\partial_z - \frac{1}{3}\partial_z^2 + \frac{1}{3}\partial_x + \partial_y - \frac{1}{3}\partial_z & \partial_y\partial_z - \frac{1}{3}\partial_z^2 \\ \partial_x\partial_z + \partial_z^2 + \partial_z & \partial_x\partial_z + \partial_z^2 \\ \partial_z^2 - \partial_x + \partial_z & 3\partial_x\partial_y + \partial_z^2 \\ \partial_x\partial_y & 0 \\ \partial_z^2 - \partial_x + \partial_z & \partial_z^2 - 3\partial_x^2 \\ \partial_x^2 & 0 \\ x\partial_z^2 - x\partial_x + \frac{3}{2}\partial_x + x\partial_z + \frac{3}{2}\partial_z + \frac{3}{2} & x\partial_z^2 + \frac{3}{2}\partial_x + \frac{3}{2}\partial_z \\ \partial_z^3 + 2\partial_z^2 + \partial_z & \partial_z^3 + \partial_x\partial_z + \partial_z^2 \end{pmatrix}.$$

The 2-step filtration $0 \leq \operatorname{tor} M \leq M$ is trivial since $M = \operatorname{tor} M$ is a torsion module (i.e., purely autonomous).

The purity filtration of M unveils the fine structure of autonomy and yields an isomorphism $\alpha : \operatorname{coker} T \to \operatorname{coker} A$ with an equivalent block triangular matrix

$$T = \left(\begin{array}{c|c|c} \partial_x & -\frac{1}{3} & 0 \\ \hline \cdot & \partial_y & \frac{1}{3} \\ \cdot & \partial_x & -\frac{1}{3} \\ \hline \cdot & \cdot & \partial_z \\ \cdot & \cdot & \partial_y \\ \cdot & \cdot & \partial_x \end{array}\right) \in D^{6\times 3},$$

i.e., $M \cong \operatorname{coker} T$ with $M_0 = 0$, $M_1 \cong \operatorname{coker} \left(\partial_x\right)$, $M_2 \cong \operatorname{coker} \begin{pmatrix} \partial_y \\ \partial_x \end{pmatrix}$, and $M_3 \cong \operatorname{coker} \begin{pmatrix} \partial_z \\ \partial_y \\ \partial_x \end{pmatrix}$, for the higher torsion modules. The isomorphism α is represented by a matrix

$$V := \begin{pmatrix} -\frac{1}{3} & -\frac{1}{3} \\ -\partial_x & -\partial_x \\ -3\partial_y\partial_z - 3\partial_y & -3\partial_y\partial_z \end{pmatrix} \in D^{3\times 2}.$$

There exists a matrix U satisfying $A = UTV$, which can now be easily computed. Finally, it is easy to see that the first generator of $\operatorname{coker} T \cong M$ is cyclic, yielding, by composition with α, an isomorphism γ from the cyclic module

$$C := D/\langle \partial_x^2 + \partial_x\partial_y, \partial_x\partial_y\partial_z, \partial_x\partial_y^2 \rangle$$

onto M. The isomorphism γ is represented by the matrix $\begin{pmatrix} 1 & 1 \end{pmatrix} \in D^{1\times 2}$ and its inverse $\gamma^{-1} : M \to C$ is represented by the matrix

$$L := \begin{pmatrix} 2x\partial_x\partial_y - \partial_x - \partial_z \\ -2x\partial_x\partial_y + \partial_x + \partial_z + 1 \end{pmatrix} \in D^{2\times 1}.$$

The easy-to-compute general solution

$$u(x,y,z) = C_1(y,z) + (x+y)C_2(z) + \bar{C}_2(z) + \frac{x^2 + 2xy + y^2}{2} C_3$$

of the simple *constant coefficient scalar* system $u_{xx} + u_{xy} = u_{xyz} = u_{xyy} = 0$ (corresponding to the relations of the cyclic module C) can now be transformed by L to the general solution $\psi = Lu$ of the complicated system[8] $A\psi = 0$.

[8]The matrices L and A act as matrices of differential operators on the sections u and ψ, respectively.

All the above examples were computed using a spectral sequence implementation in the GAP package homalg [Bar10], [hpa10]. The algorithms are described in [Bar]. See also [Qua10c] for an implementation of QUADRAT's recent approach to the purity filtration, which does not make use of spectral sequences.

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