LQ-optimal control for a class of time-varying coupled PDEs-ODEs system

Amir Alizadeh Moghadam, Ilyasse Aksikas, J. Fraser Forbes, Stevan Dubljevic

***Abstract*— This contribution addresses the development of a Linear Quadratic Regulator (LQR) for a set of time-varying hyperbolic PDEs coupled with a set of time-varying ODEs through the boundary. The approach is based on an infinite-dimensional *Hilbert* state-space realization of the system and *operator Riccati equation* (ORE). In order to solve the optimal control problem, the ORE is converted to a set of *differential* and *algebraic matrix Riccati equations*. The feedback gain can then be found by solving the resulting *matrix Riccati equations*. The control policy is applied to a system of continuous stirred tank reactor (CSTR) and a plug flow reactor (PFR) in series and the controller performance is evaluated by numerical simulation.**

I. INTRODUCTION

Mathematical modelling of chemical processes commonly involves lumping assumption to covert to ordinary differential equations (ODEs); however, this cannot be the case for some unit operations, such as plug-flow reactors and packed bed columns in which space variations are important. These systems are called distributed parameter systems and modelled by partial differential equations (PDEs). Some complicated chemical systems involve both lumped and distributed parameter parts (LPS-DPS) and are described by a combination of ODEs and PDEs. For instance, a jacket-equipped fixed-bed reactor is lumped on the jacket side and is distributed on the reactor side.

Despite the importance and inherent complexities in the structure of composite lumped and distributed parameter systems, research in the area of feedback control for these systems is scarce. By using the maximum principle for parabolic partial differential equations, Wang ([1]) derived sufficient conditions for stability and asymptotic stability for a mixed parabolic PDE and ODE system. Tzafestas ([2]) used classical calculus of variations to solve the optimal control problem for non-linear, mixed lumped and distributed parameter systems. He also found necessary optimality conditions for the optimal final-value problem for these systems by applying Green's identity and functional analysis techniques ([3]). Dynamic programming was used in [4] and [5] to solve discrete-time optimal feedback control problem for a class of linear composite systems.

LQR control plays a significant role in optimal control literature. The main objective of this control policy is to regulate a linear system by minimizing a quadratic performance index. In solving a LQR problem for an infinite-dimensional (distributed) system, two common methods are available in the literature. The first approach is based on frequency domain description and known as *spectral factorization*. In this method the control law is obtained via solving an *operator Diophantine equation* ([6]). This technique is applied in [7] to control the temperature and the concentration in a plug flow reactor. The second methodology involves solving an *operator Riccati equation* for a given state-space model ([8]). This method was used for a particular class of hyperbolic PDEs ([9]). This work was then extended to a more general class of hyperbolic system by using an infinite-dimensional *Hilbert* state-space setting with distributed input and output([10]).

In our previous work [11], LQR control for a set of coupled linear hyperbolic PDEs and ODEs was achieved by solving an *operator Riccati equation* for the infinite-dimensional *Hilbert* state-space description of the system. The designed control policy was applied to a process containing a CSTR in series with a PFR and tested via numerical simulation. This work assumed time-invariant systems. The aim of this paper is to generalize the previous work to the time-varying case. The time-varying case is an important problem in chemical reactor operations; due to catalyst deactivation phenomenon. Several models for catalyst deactivation were proposed in [12] and the simple exponential decay model is used in this paper.

The paper is organized as follows. In section 2, general formulation for a system including a set of composite linear time-varying first-order hyperbolic PDEs and linear ODEs is given. The system is described as an infinite-dimensional *Hilbert* state-space. A state transformation is then used to convert the related boundary condition to a homogeneous one. Section 3 focuses on formulating and solving the LQR problem for the mentioned system. To this end, the *operator Riccati equation* is computed. This results in four *matrix Riccati equations*, which should be solved to obtain state feedback gain. In order to evaluate the performance of the proposed method, in section 4, the designed control policy is applied to a system containing a CSTR and a PFR in series in which a time-varying Van de Vusse reaction is taking place . First, the system is linearized around the equilibrium point. Then, the feedback control gain is obtained by solving the related *matrix Riccati equations*. Finally, the designed control policy is applied to the original non-linear system and simulation results are discussed.

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II. PROBLEM STATEMENT

This paper focuses on composite linear time-varying LPS-DPS, where the control variable affects the boundary of the distributed parameter system directly or indirectly, and through the lumped parameter system. In addition, the DPS is assumed to be described by a set of first-order hyperbolic PDEs. The general mathematical formulation for these systems is as follows:

$$\frac{\partial x_d}{\partial t}(t,z) = V\frac{\partial x_d}{\partial z}(t,z) + M(t,z)x_d(t,z) + B_{d0}(t,z)u(t) \tag{1}$$

$$\frac{dx_l}{dt}(t) = A(t)x_l(t) + B(t)u(t) \tag{2}$$

with the following boundary and initial conditions:

$$x_d(t,0) = x_l(t) \tag{3}$$

$$x_d(0,z) = x_{d,0}(z) \tag{4}$$

$$x_l(0) = x_{l,0} \tag{5}$$

where, $x_d(.,t) \in L_2(0,1)^n$ and $x_l(t) \in \mathbb{R}^n$ denote the state variables for the distributed and the lumped parameter systems, respectively, $z \in [0,1]$ is the spatial coordinate, $t \in [0,\infty]$ is the time, $u(t) \in \mathbb{R}^m$ is the input variable, $V = -\upsilon I \in \mathbb{R}^{n\times n}$ with $\upsilon > 0$ is a symmetric matrix, $M(t,z)$ and $B_{d0}(t,z)$ are real continuous space and time-varying matrices, $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$ are real time-varying matrices, $x_{d,0}$ is a real continuous space varying vector, and $x_{l,0}$ is a constant vector.

The above system can be stated as an infinite-dimensional state-space in the *Hilbert* space $\mathscr{H} = L_2(0,1)^n$ ([8]):

$$\dot{x}_d(t) = \mathscr{A}(t)x_d(t) + B_d(t)u(t) \tag{6}$$

$$\dot{x}_l(t) = A(t)x_l(t) + B(t)u(t) \tag{7}$$

$$\mathscr{B}x_d(t) = x_l(t) \tag{8}$$

Here $\mathscr{A}(t)$ is a linear time-varying operator defined as:

$$\mathscr{A}(t)h(z) = V\frac{dh(z)}{dz} + M(t,z)h(z) \tag{9}$$

where $h(z)$ is a smooth function on $[0,1]$, with the following domain:

$$D(\mathscr{A}(t)) = \{h(z) \in \mathscr{H} : h(z) \text{ and } \frac{dh(z)}{dz} \text{ are abs. cont., and } \frac{dh(z)}{dz} \in \mathscr{H}\} \tag{10}$$

$\mathscr{B}$ is a linear boundary operator defined as:

$$\mathscr{B}h(z) = h(0) \tag{11}$$

$$D(\mathscr{B}) = \{h(z) \in \mathscr{H} : h(z) \text{ is abs. cont.}\} \tag{12}$$

$B_d(t)$ is given by $B_d(t) = B_{d0}(t,z)I$, where I is the identity operator.

The infinite-dimensional state-space system (6) to (8) with inhomogeneous boundary condition can be transformed to a new system with homogenous boundary condition using *boundary control transformation* (see [8] and [13]). We assume that there is a function $\mathfrak{B}(z)$ such that for all $x_l(t)$, $\mathfrak{B}x_l(t) \in D(\mathscr{A}(t))$ and:

$$\mathscr{B}\mathfrak{B}(z)x_l(t) = x_l(t) \tag{13}$$

By assuming that $x_l(t)$ is sufficiently smooth and using the state transformation $\omega(t) = x_d(t) - \mathfrak{B}(z)x_l(t)$ ([8]), we have:

$$\dot{\omega}(t) = \dot{x}_d - \mathfrak{B}(z)\dot{x}_l$$

Then:

$$\dot{\omega}(t) = \mathscr{F}(t)\omega(t) + \mathscr{A}(t)\mathfrak{B}(z)x_l(t) + B_d(t)u(t) - \mathfrak{B}(z)\dot{x}_l$$
$$\omega(0) = \omega_0 \tag{14}$$

where $\omega_0 = x_{d,0} - \mathfrak{B}(z)x_{l,0} \in D(\mathscr{F})$ and:

$$\mathscr{F}(t)h(z) = \mathscr{A}(t)h(z)$$

The domain of $\mathscr{F}(t)$ is defined as:

$$D(\mathscr{F}(t)) = D(\mathscr{A}(t)) \cap ker(\mathscr{B}) = \{h(z) \in \mathscr{H} : h(z) \text{ and } \frac{dh(z)}{dz} \text{ are abs. cont., } \frac{dh(z)}{dz} \in \mathscr{H}, \text{ and } h(0) = 0\} \tag{15}$$

By combining (7) and (14) we obtain the new infinite-dimensional *Hilbert* state-space representation of the time-varying DPS-LPS as:

$$\begin{bmatrix}\dot{\omega}(t)\\ \dot{x}_l(t)\end{bmatrix} = \begin{bmatrix}\mathscr{F}(t) & \mathscr{A}(t)\mathfrak{B}(z) - \mathfrak{B}(z)A(t)\\ 0 & A(t)\end{bmatrix}\begin{bmatrix}\omega(t)\\ x_l(t)\end{bmatrix} + \begin{bmatrix}B_d(t) - \mathfrak{B}(z)B(t)\\ B(t)\end{bmatrix}u(t)$$
$$\omega(0) = \omega_0, x_l(0) = x_{l,0} \tag{16}$$

We define state and output variables of the above system as $x(t) = [\omega(t), x_l(t)]^T$ and $y(z,t) = [y_d(z,t), y_l(t)]^T$, where $y_d(.,t) \in \mathscr{Y} := L_2(0,1)^p$ is the output variable for the distributed parameter system and $y_l(t) \in \mathbb{R}^p$ is the output variable for the lumped parameter system. Then the output equation will be

$$y(t) = Cx(t) \tag{17}$$

where C is given by $C = C_0 I$ with I is the identity operator. Here C_0 is given by $C_0 = \begin{bmatrix}S_0 & 0\\ 0 & S_0\end{bmatrix}$ with $S_0 \in \mathbb{R}^{p\times n}$.

In [14], it is proven that given $V < 0$, operator $\mathscr{F}(t)$ generates an exponentially stable C_0-semigroup. Therefore, If matrix $A(t)$ is stable, operator $\begin{bmatrix}\mathscr{F}(t) & \mathscr{A}(t)\mathfrak{B}(z) - \mathfrak{B}(z)A(t)\\ 0 & A(t)\end{bmatrix}$ provides a stable C_0-semigroup.

III. OPTIMAL CONTROL DESIGN

In this section we are interested in LQR control synthesis for the time-varying DPS-LPS system according to the infinite-dimensional state-space representation of (16). The design is based on the minimization of an infinite-horizon quadratic objective function that requires the solution of an *operator Riccati equation* ([8], [15]). Solution of the *operator Riccati equation* for the time-varying DPS-LPS results in a set of *algebraic, ordinary differential and partial differential matrix Riccati equations*. The optimal feedback gain can then be found by solving the equivalent *matrix Riccati equations*.

Let us consider the following quadratic objective function:

$$J(x_0, u) = \int_0^{\infty} (\langle Cx(t), PCx(t)\rangle + \langle u(t), Ru(t)\rangle)dt \tag{18}$$

where $x_0 \in \mathscr{H}$ is an initial condition, $P = P_0 I \in \mathcal{L}(\mathscr{Y})$, $P_0 = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathbb{R}^{2p\times 2p}$ is a positive semi-definite symmetric matrix, and $R \in \mathbb{R}^{m\times m}$ is a positive symmetric matrix. The minimization of the above objective function subject to the system of (16) results in solving the following *operator Riccati equation* ([14] and the references therein):

$$\begin{aligned}[\dot{Q}_0 + \mathcal{A}(t)^* Q_0 + Q_0 \mathcal{A}(t) + C^* PC- \\ Q_0 \mathcal{B}(t) R^{-1} \mathcal{B}(t)^* Q_0] x = 0\end{aligned} \tag{19}$$

where $\mathcal{A}(t) = \begin{bmatrix} \mathscr{F}(t) & \mathscr{A}(t)\mathfrak{B}(z) - \mathfrak{B}(z)A(t) \\ 0 & A(t) \end{bmatrix}$, $\mathcal{B}(t) = \begin{bmatrix} B_d(t) - \mathfrak{B}(z)B(t) \\ B(t) \end{bmatrix}$ and $Q_0 \in \mathcal{L}(\mathscr{H})$ is non-negative self-adjoint operator. The above *operator Riccati equation* has a unique solution Q_0. The minimum cost function is given by $J(x_0, u_{opt}) = \langle x_0, Q_0 x_0\rangle$ (see [14]). For any initial condition $x_0 \in \mathscr{H}$ the unique optimal control variable u_{opt}, which minimizes the objective function of (19), is obtained on $t \geq 0$ as:

$$u_{opt}(t) = K(t)x(t) \tag{20}$$

where

$$K(t) = -R^{-1}\mathcal{B}(t)^* Q_0(t) \tag{21}$$

Under this condition, $\mathcal{A}(t) + \mathcal{B}(t)K(t)$ generates an exponentially stable C_0-semigroup ([14]).

Let the solution of the *operator Riccati equation* (19) be:

$$Q_0(t) := \begin{bmatrix} \Phi(t,z)I & 0 \\ 0 & \Psi(t)I \end{bmatrix} \tag{22}$$

where $\Phi(t,z), \Psi(t) \in \mathbb{R}^{n\times n}$ are positive self-adjoint matrices and I is the identity operator. By substituting for $\mathcal{A}(t)$, $\mathcal{B}(t)$, C, and Q_0 in (19), we have:

$$\begin{aligned}\dot{\Phi} + \mathscr{F}^*\Phi + \Phi\mathscr{F} + S_0^* P_{11} S_0 - \\ \Phi(B_d - \mathfrak{B}B)R^{-1}(B_d - \mathfrak{B}B)^*\Phi = 0\end{aligned} \tag{23}$$

$$\begin{aligned}\Phi(\mathscr{A}\mathfrak{B} - \mathfrak{B}A) + S_0^* P_{12} S_0 - \\ \Phi(B_d - \mathfrak{B}B)R^{-1}B^*\Psi = 0\end{aligned} \tag{24}$$

$$\begin{aligned}(\mathscr{A}\mathfrak{B} - \mathfrak{B}A)^*\Phi + S_0^* P_{21} S_0 - \\ \Psi B R^{-1}(B_d - \mathfrak{B}B)^*\Phi = 0\end{aligned} \tag{25}$$

$$\dot{\Psi} + A^*\Psi + \Psi A + S_0^* P_{22} S_0 - \Psi B R^{-1} B^* \Psi = 0 \tag{26}$$

Equation (23) is a *differential matrix Riccati equation*. We assume that the matrix $V = -\upsilon I, \upsilon > 0$ is diagonal with identical entries. In this condition, (23) can be solved by the following set of PDEs (see [14] for proof):

$$\begin{aligned}\frac{\partial\Phi}{\partial t} = -V\frac{d\Phi}{dz} + M^*\Phi + \Phi M + S_0^* P_{11} S_0 - \\ \Phi(B_d - \mathfrak{B}B)R^{-1}(B_d - \mathfrak{B}B)^*\Phi \\ \Phi(1) = 0\end{aligned} \tag{27}$$

Equation (26) is an *ordinary differential matrix Riccati equation*, which can be integrated over time to find Ψ. Equation (25) is the adjoint of (24) and, therefore, these two equations are the same. Equation (24) can be satisfied by using elements of matrix P_{12} such that matrix P_0 remains positive semi-definite. We can derive the following equation from (24):

$$S_0^* P_{12} S_0 = -\Phi M + \Phi A + \Phi(B_d - \mathfrak{B}B)R^{-1}B^*\Psi \tag{28}$$

In order to get the above equation, first we obtain $\mathfrak{B}(z)$ from (13) as:

$$\mathscr{B}\mathfrak{B}(z) = I \tag{29}$$

Then:

$$[\mathfrak{B}(z)]_{z=0} = I, \mathfrak{B}(z) = I \in D(\mathscr{A}) \tag{30}$$

By using (9) we have:

$$\mathscr{A}(t)\mathfrak{B}(z) = V\frac{d\mathfrak{B}(z)}{dz} + M(t,z)\mathfrak{B}(z) \tag{31}$$

Let us substitute expression for $\mathfrak{B}$ into (31), which yields:

$$\mathscr{A}\mathfrak{B} = M\mathfrak{B} \tag{32}$$

By substituting for $\mathscr{A}\mathfrak{B}$ in (24), we obtain (28).

Therefore, the solution procedure for the LQR problem can be stated as:

- Choose weighting matrices P_{11}, P_{22}, and R
- Find $\Psi(t)$ and $\Phi(t,z)$ via solving (26) and (27) (*differential matrix Riccati equations*)
- Obtain P_{12} from (28) and check whether matrix P_0 is positive semi-definite or not
- In the case that P_0 is not positive semi-definite choose new P_{11} and P_{22} and resolve (26) and (27)
- Calculate the feedback gain from (21)

It should be noticed that in the case of state LQR where $S_0 = I$ the left hand side of (28) reduces to P_{12}.

IV. CASE STUDY

In this section we consider a CSTR-PFR configuration shown in Fig. 1 as a composite lumped and distributed parameter system. This reactor configuration is recommended for some types of chemical reactions (e.g., [16] and [17]) and may be used to carry out Van de Vusse reaction to achieve maximum conversion to the desired product ([17]). Here, we assume reactions and kinetics:

$$A \longrightarrow B, \quad -r_1 = k_1 e^{-E_1/RT} C_A \tag{33}$$

$$B \longrightarrow C, \quad -r_2 = k_2 e^{-E_2/RT} C_B \tag{34}$$

$$2A \longrightarrow D, \quad -r_3 = k_3 e^{-E_3/RT} C_A^2 \tag{35}$$

where: k_1, k_2 and k_3 are pre-exponential constants; E_1, E_2 and E_3 are the activation energy and R is the universal gas constant. The exothermic reactions take place in both CSTR and PFR and component B is the desired component. The reaction kinetics are considered to be time-varying according to the following exponential decay model (see [12]):

$$k_i = k_{0,i} + k_{1,i} e^{-\alpha_i t} \tag{36}$$

where subscripts $i = 1, 2, 3$ denote number of reactions and $k_{0,i}$, $k_{1,i}$ and α_i are the decay model parameters.

The objective is to control the concentration of the components and the temperature within both reactors by using inlet flow rate (F_{in}) and cooling rate from the CSTR (Q) as manipulated variables. With the assumptions of negligible diffusion in the PFR, perfect level control in the CSTR, no transportation lags in the connecting lines, constant fluid velocity in the PFR with respect to spatial coordinate and constant physical properties, the mathematical model of the system will be:

$$\frac{dC_A}{dt} = \frac{F_{in}}{V_s}(C_A^{in} - C_A) - k_1 e^{-E_1/RT} C_A - k_3 e^{-E_3/RT} C_A^2 \tag{37}$$

$$\frac{dC_B}{dt} = -\frac{F_{in}}{V_s} C_B + k_1 e^{-E_1/RT} C_A - k_2 e^{-E_2/RT} C_B \tag{38}$$

$$\frac{dT}{dt} = \frac{1}{\rho c_p}[k_1 e^{-E_1/RT} C_A(-\Delta H_1) + k_2 e^{-E_2/RT} C_B(-\Delta H_2) + k_3 e^{-E_3/RT} C_A^2(-\Delta H_3)] + \frac{F_{in}}{V_s}(T_{in} - T) + \frac{Q}{\rho c_p V_s} \tag{39}$$

$$\frac{\partial C_A^p}{\partial t} = -\upsilon \frac{\partial C_A^p}{\partial z} - k_1 e^{-E_1/RT} C_A^p - k_3 e^{-E_3/RT} {C_A^p}^2 \tag{40}$$

$$\frac{\partial C_B^p}{\partial t} = -\upsilon \frac{\partial C_B^p}{\partial z} + k_1 e^{-E_1/RT} C_A^p - k_2 e^{-E_2/RT} C_B^p \tag{41}$$

$$\frac{\partial T_p}{\partial t} = -\upsilon \frac{\partial T_p}{\partial z} + \frac{k_1}{\rho c_p} e^{-E_1/RT} C_A^p(-\Delta H_1) + \frac{k_2}{\rho c_p} e^{-E_2/RT} C_B^p(-\Delta H_2) + \frac{k_3}{\rho c_p} e^{-E_3/RT} {C_A^p}^2(-\Delta H_3) \tag{42}$$

$$C_A^p(t, 0) = C_A \tag{43}$$

$$C_B^p(t, 0) = C_B \tag{44}$$

$$T_p(t, 0) = T \tag{45}$$

where: C_A and C_B are the concentration of the components A and B in the CSTR, respectively; T is the temperature in the CSTR; C_A^p and C_B^p are the concentration of the components A and B in the PFR, respectively; T_p is the temperature in the PFR; $z \in [0, L]$ is the spatial coordinate; $t \in [0, \infty]$ is the time; F_{in}, C_A^{in} and T_{in} are the volumetric flow-rate, concentration and temperature of the feed to the CSTR; V_s and V_p are the volumes of the CSTR and the PFR, respectively; υ is the fluid velocity in the PFR which is given by $\upsilon = \frac{F_{in} L}{V_p}$, ΔH_1, ΔH_2 and ΔH_3 are the heat of reaction for the reactions 1, 2 and 3, respectively; and ρ and c_p are the average fluid density and specific heat.

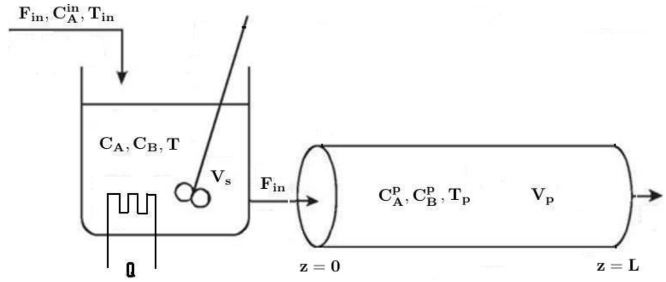

Fig. 1. CSTR-PFR system

TABLE I
MODEL PARAMETERS

Parameter	Value
$k_{0,1}$	$225.2250 \times 10^6 \quad sec^{-1}$
$k_{0,2}$	$225.2250 \times 10^6 \quad sec^{-1}$
$k_{0,3}$	$1.583 \times 10^6 \quad sec^{-1}$
$k_{1,1}$	$25.025 \times 10^6 \quad sec^{-1}$
$k_{1,2}$	$25.025 \times 10^6 \quad sec^{-1}$
$k_{1,3}$	$1.759 \times 10^5 \quad sec^{-1}$
$\alpha_1, \alpha_2, \alpha_3$	$1.389 \times 10^{-3} \quad sec^{-1}$
$F_{in,ss}$	$174.845 \times 10^{-6} \quad m^3/sec$
Q_{ss}	$-1.36 \quad kJ/sec$
C_A^{in}	$5.1 \quad kmol/m^3$
T_{in}	$403.15 \quad K$
ΔH_1	$-4200 \quad kJ/kmol$
ΔH_2	$-11000 \quad kJ/kmol$
ΔH_3	$-41850 \quad kJ/kmol$
E_1/R	$9758.3 \quad K$
E_2/R	$9758.3 \quad K$
E_3/R	$8560.0 \quad K$
V_s	$0.01 \quad m^3$
V_p	$0.005 \quad m^3$
ρ	$934.2 \quad kg/m^3$
c_p	$3.01 \quad kJ/kgK$

The model parameters used in this case study are given in Table I. In order to find the equilibrium condition for the system, the modelling equations (37) to (45) have been solved at steady-state in gPROMS® ([18]). Simulation yields the steady-state values for C_A, C_B and T of 2.71 $kmol/m^3$, 1.07 $kmol/m^3$ and 409.79 K, respectively. The steady-state profiles for C_A^p and C_B^p are shown in Fig. 2 and the steady-state profile for T_p is shown in Fig. 3. The model equations can be linearized around the steady-state condition to describe the system as the general formulation of (1) to (5).

A. LQR Control Design

We use the proposed optimal control policy to control the concentration of the components and also the temperature in both CSTR and PFR. S_0 is selected to be the identity matrix

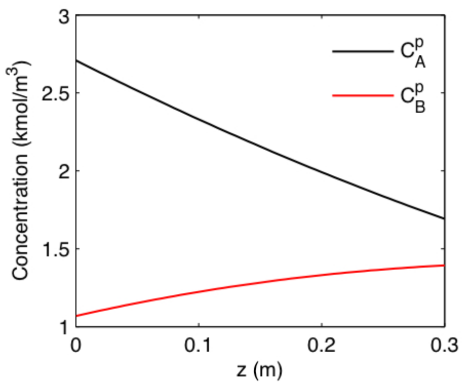

Fig. 2. Steady-state concentration profiles in the PFR

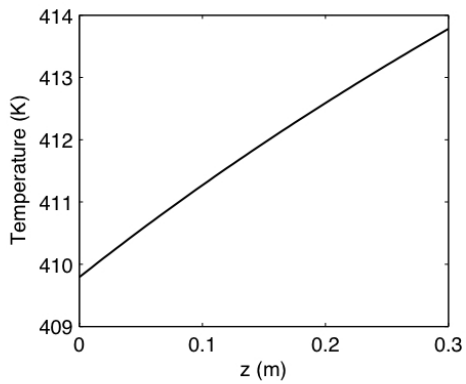

Fig. 3. Steady-state temperature profile in the PFR

to yield a state LQR problem. In order to solve the LQR control problem for this system, we follow the procedure discussed in Section III. The *matrix Riccati equations* (26) and (27) are solved numerically in gPROMS. By choosing $P_{11} = 100I$, $P_{22} = 250I$ and $R = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.0001 \end{bmatrix}$ the matrix P_0 is positive semi-definite. The feedback gain can be obtained using (21).

B. Simulation Results

In order to assess the performance of the control policy, the designed feedback gain is implemented to the original non-linear system (37) to (45). The coupled non-linear and time-vatying LPS-DPS is solved using *orthogonal collocation on finite element* method in gPROMS. We use $C_A(0) = 10\ kmol/m^3$, $C_B(0) = 10\ kmol/m^3$, $T(0) = 406\ K$, $C_A^p(0,z) = 1\ kmol/m^3$, $C_B^p(0,z) = 1\ kmol/m^3$ and $T_p(0,z) = 403\ K$ as initial conditions. Open-loop and closed-loop responses for the concentration of the components and the temperature in the CSTR are compared in Fig. 4 and Fig. 5, respectively. As can be seen, the closed-loop response is significantly better than the open-loop response. The response time of the closed-loop system is more than two times faster and the deviations from the operating condition is much less. In Fig. 6 and Fig. 7 open-

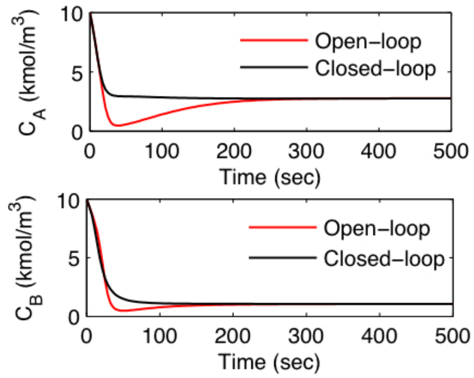

Fig. 4. Concentration responses in the CSTR

loop and closed-loop responses for the concentration of the components and temperature at the outlet of the PFR are shown. As can be observed, the inverse-response for the open-loop concentrations have disappeared and again, the closed-loop responses are much faster than open-loop ones. Finally, the variations of the control inputs are shown in Fig. 8. The control efforts are not particularly aggressive and are physically realizable.

V. CONCLUSIONS

In this work LQR control problem for a class of time-varying composite lumped and distributed parameter system is formulated and solved. In this mixed system, the lumped parameter system interacts with the hyperbolic distributed parameter system through its boundary. The control variables can affect the boundary of the distributed parameter system directly or indirectly, and through the lumped parameter system. The LQR control problem is formulated based on an infinite-dimensional state-space representation of the system, which is obtained via a state transformation from the original system using the idea of boundary control problem. The solution of the LQR control problem is achieved by solving the *matrix Riccati equations* that results from the *operator Riccati equation* of the *infinite-dimensional* state-space representation. The designed optimal control policy was implemented on an interacting CSTR-PFR system with time-varying reaction kinetics. The performance of the controller was assessed by implementing the controller on the original non-linear system and resulted in a high performance closed-loop system.

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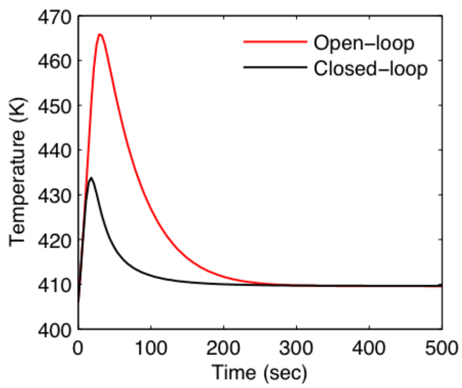

Fig. 5. Temperature response in the CSTR

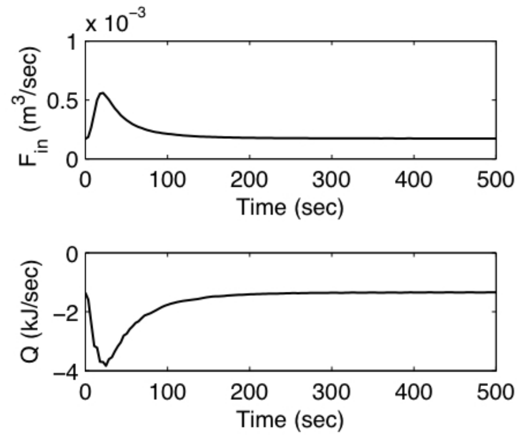

Fig. 8. Control inputs

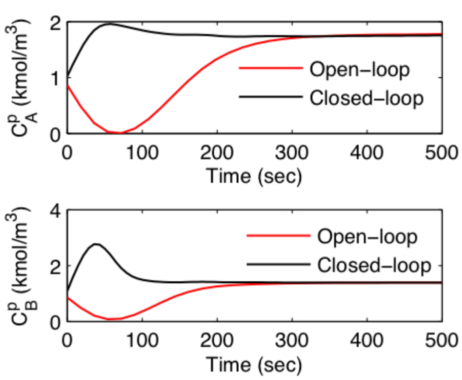

Fig. 6. Concentration responses at the outlet of the PFR

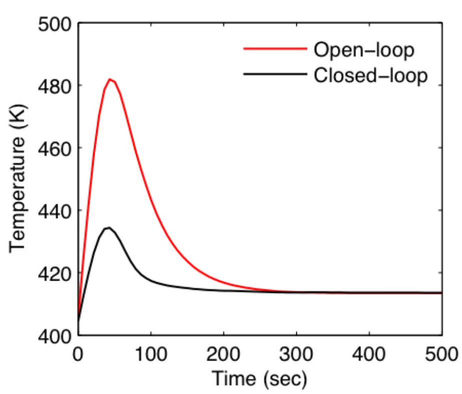

Fig. 7. Temperature response at the outlet of the PFR

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