Random Walk on a Rooted, Directed Husimi Cactus

Tamás Kalmár-Nagy

***Abstract*— The objective of this paper is to further explore the connection between the random Fibonacci series and a random walk on the so-called triangular Husimi cactus. Various statistical properties of this random walk are computed.**

I. INTRODUCTION

Time delays play an important role in fields as diverse as biology [4], population dynamics [5], neural networks [6], and lasers [7]. Delay effects can also be exploited to control nonlinear systems ([8], [9]). Stability and bifurcation analysis of simple systems with deterministic delays is a fairly mature field. This is not the case when many components are coupled through random delays.

A ubiquitous, technologically relevant example of such systems is Networked Control Systems, where information between sensors, actuators and controllers are transmitted through a shared communication backbone. The salient feature of these systems is the coupling of dynamic components via the underlying communication network. Networked Control Systems have proven useful in diverse areas such as teleoperation [10], mobile sensor networks [12], telesurgery [11], collaborative haptics [13], and control of vehicles [14]. Because of the unpredictability of network traffic, the arrival time of signals can only be characterized by a probability distribution. The main drive of this research is to understand how the shape and parameters of the signal delay distribution influences stability of networked control systems. This paper should be considered as an exploratory step to establish a possible connection between stability properties of random systems, graph theory and random matrix products. In particular, initial study of digital control systems with random time delays gave rise to a discrete-time jump linear systems, where the transition jumps are modeled with an underlying finite-state Markov chain ([16], [17], [18]). Usually, the underlying distributions are very complicated. To gain insight into the underlying random process, we focus our attention to the study of a simple, but characteristic problem of this class.

One simple, non-trivial such model is the so-called random Fibonacci sequence. This is a second-order difference equation where the coefficient matrix is randomly chosen (from a set of two integer valued matrices) at every time step. Previously ([2]) the equivalence was established between the random Fibonacci recurrence and a random walk on a self-similar graph whose vertices are the visible points of the plane. This graph, the so-called Husimi cactus, has been studied in the context of statistical physics. Here we study the simple random walk on the rooted Husimi cactus.

T. Kalmár-Nagy is with the Department of Aerospace Engineering, Texas A&M University, College Station, 77843 Texas, USA mtns2010@kalmarnagy.com

II. LINEAR CONTROL SYSTEMS WITH RANDOM ACTUATION DELAYS

We consider a simple model of a linear control system with random actuation delays (Fig. 1). The random delays are due to communication over a network.

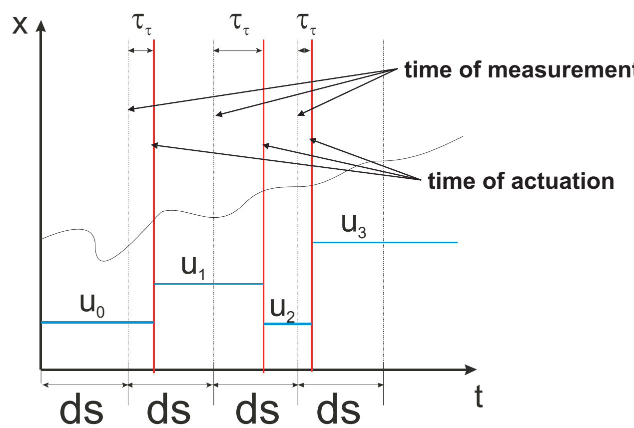


Fig. 1. Control system with random actuation delays.

The structure of the control system takes the usual form

$$\dot{x} = Ax + Bu \quad (1)$$

$$y = Cx, \quad (2)$$

where the state evolves continuously. The control signal is assumed to be proportional to the output. The sampling time ds specifies the discrete time instants $s_i = ids$ at which the state of the system is sampled. The actuation times would ideally be the same as the sampling times, but due to network delays these become

$$t_i = s_i + \tau_i \quad (t_0 = 0, \quad \tau_0 = 0)\,. \quad (3)$$

Here we also assume that $\tau_i < ds$. While this model is simplistic, the resulting class of problems (jump linear systems) have been shown to play an important role in describing systems with random time delays. The mapping between values of the state and control input at consecutive actuation times is given by ([3])

$$\begin{pmatrix} x \\ u \end{pmatrix}_{i+1} = \underbrace{\begin{pmatrix} C(\tau_{i+1}, \tau_i) & D(\tau_{i+1}, \tau_i) \\ E(\tau_i) & F(\tau_i) \end{pmatrix}}_{\mathcal{A}_i} \begin{pmatrix} x \\ u \end{pmatrix}_i \quad (4)$$

ISBN 978-963-311-370-7

where

$$C\left(\tau_{i+1}, \tau_i\right) = e^{A\left(ds+\tau_{i+1}-\tau_i\right)}, \tag{5}$$

$$D\left(\tau_{i+1}, \tau_i\right) = e^{A\left(ds+\tau_{i+1}-\tau_i\right)} \int\limits_0^{ds+\tau_{i+1}-\tau_i} e^{-As} ds B, \tag{6}$$

$$E\left(\tau_i\right) = e^{A\left(ds-\tau_i\right)}, \tag{7}$$

$$F\left(\tau_i\right) = e^{A\left(ds-\tau_i\right)} \int\limits_0^{ds-\tau_i} e^{-As} ds B. \tag{8}$$

This map relates the state and control input at step to their values at the previous step. The nth element of the sequence $\left\{ \begin{pmatrix} x \\ u \end{pmatrix}_i, i = 1, 2, .. \right\}$ can be explicitly written as

$$\begin{pmatrix} x \\ u \end{pmatrix}_n = \prod_{i=0}^{n} \mathcal{A}_i \begin{pmatrix} x \\ u \end{pmatrix}_0. \tag{9}$$

The stability of the system (the rate of convergence/divergence) is determined by the growth/decay of the associated infinite random matrix product. Note that in general $\mathcal{A}_{i+1}$ is dependent of $\mathcal{A}_i$. To account for this dependence, the distribution of these random matrices are described over an associated Markov chain. These systems are also known as jump linear systems, constituting a class of hybrid systems. While for a specified distribution of network delays Monte-Carlo simulations could provide information about the stability of the system, our goal is to understand the underlying principles that govern this class of problems. Therefore, in the following we focus on studying one of the simplest jump linear system, namely the random Fibonacci sequence. By establishing connections with graph theory and statistical physics, it is our hope that these tools will bring a better understanding of stability properties of linear systems over Markov chains and consequently of problems with random time delays.

III. THE RANDOM FIBONACCI SEQUENCE AND THE HUSIMI CACTUS

One of the simplest examples of random matrix products is the so-called random Fibonacci series studied by Viswanath [15]

$$x_n = x_{n-1} \pm x_{n-2}, \quad x_0 = x_1 = 1, \tag{10}$$

where different signs are chosen independently with probability $1/2$ at each step. Viswanath [15] utilized a powerful combination of random matrix theory and interval arithmetic to show that the sequence almost surely diverges with an exponent of 1.132, i.e. $\nu = \lim_{n\to\infty} \sqrt[n]{|x_n|} \cong 1.132$. Embree and Trefethen [1] numerically investigated the recurrence $x_{n+1} = x_n \pm \beta x_{n-1}$. The measure of divergence for the random Fibonacci series can be expressed as a path average by considering all the possible values x_n can attain. The exponent ν (Viswanath's number) can then be formally expressed as the limit

$$\nu = \lim_{n\to\infty} \left(\prod_{0\neq x} |x|^{1/m} \right)^{1/n} = \lim_{n\to\infty} \left| \prod_{0\neq x} x \right|^{1/(nm)}, \tag{11}$$

where the (geometric) mean is taken over all the m possible values of x at step n (here $m = 2^{n-2}$). Kalmár-Nagy [2] has shown that the random Fibonacci sequence is equivalent to a random walk on the visible points of the first quadrant of the integer lattice $\mathbb{Z}^2$ $(i, j > 0,\ \gcd(i, j) = 1)$. Moreover, it was shown that this random walk takes places on an infinite, rooted, directed Husimi cactus with a Euclidean metric (Fig. 2).

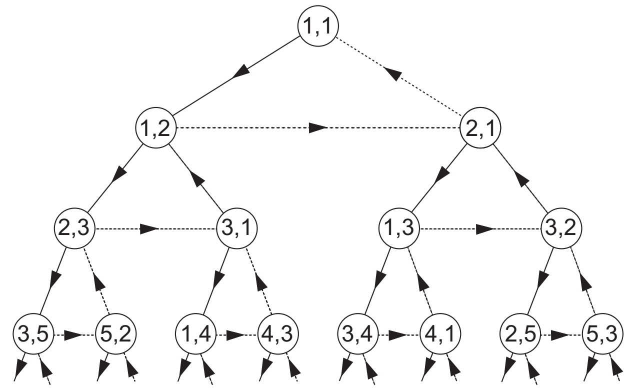


Fig. 2. Rooted Husimi cactus. The nodes are the visible points (i, j), $\gcd(i, j) = 1$.

IV. RANDOM WALK ON THE ROOTED HUSIMI CACTUS

Here we are interested in deriving some properties of the random walk on the ternary Husimi graph. In the following, we only pay attention to the topological distance (the length of the shortest directed path between vertices). Distances from the root are shown in Figure 3. Additionally, we characterize nodes as type A $(i < j)$ or type B $(i \geq j)$.

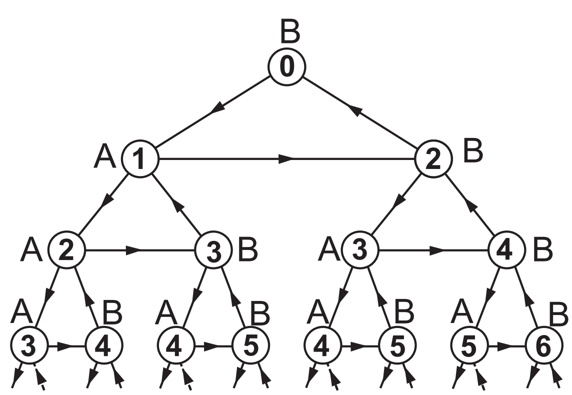


Fig. 3. Rooted Husimi cactus with graph theoretical distance.

The following statements provide significant information about the structure of the cactus.

Proposition 1: The number of type A and B points that are distance $m > 1$ away from the root are F_m and F_{m-1}, where F_m denotes the m-th Fibonacci number ($F_0 = F_1 = 1$).

For a nice proof of this statement, see [20].

Corollary 2: The number of length-m ($m > 1$) simple paths (paths that connect distinct vertices) in the Husimi cactus is F_{m+1}, simply the total number of vertices at distance m from the root ($F_{m-1} + F_m = F_{m+1}$).

The probability of the random walker visiting a site at distance i at time n is simply the ratio of the number of length-i walks and the total number of walks of length n. The main mathematical tool here will be generating functions [22]. First we derive the generating function $g(x,i)$, where i is the distance from the root.

Initially (at timestep $n = 0$) the walker is at the root (at distance $x = 0$). The number assigned to a site at time n is the sum of the numbers at its neighboring sites at time $n-1$. We will now translate these rules into the language of generating functions. The number at the i-th site at time n is represented by the coefficient of x^n in the function $g(x,i)$, or $[n]\, g(x,i)$ in shorthand. For the root

$$[n]\, g(x,0) = [n-1]\, g(x,2) \tag{12}$$

holds, while for $i > 0$

$$[n]\, g(x,i) = [n-1]\, g(x,i-1) + [n-1]\, g(x,i+2)\,. \tag{13}$$

Also the generating functions should satisfy the initial conditions

$$[0]\, g(x,i) = \delta_{0i}, \tag{14}$$

where δ is the Kronecker-delta. Since the constant terms of the generating functions (for $i > 0$) are zero,

$$[n]\, g(x,i) = [n-1]\, \frac{g(x,i)}{x}. \tag{15}$$

Therefore

$$\frac{g(x,i)}{x} = g(x,i-1) + g(x,i+2)\,. \tag{16}$$

The ansatz $g(x,i) = x^i f\left(x^3\right)^{i+1}$ is substituted to yield

$$x^{i-1} f\left(x^3\right)^{i+1} = x^{i-1} f\left(x^3\right)^{i} + x^{i+2} f\left(x^3\right)^{i+3}, \tag{17}$$

and finally

$$f\left(x^3\right) = 1 + x^3 f\left(x^3\right)^3. \tag{18}$$

With a new variable $y = x^3$ we have

$$f(y) = 1 + y f(y)^3. \tag{19}$$

Gessel and Xin [21] showed that

$$f(y) = \sum_{n=0}^{\infty} \frac{1}{3n+1} \binom{3n+1}{n} y^n = \tag{20}$$

$$\frac{2}{\sqrt{3y}} \sin\left(\frac{1}{3} \arcsin \frac{3\sqrt{3y}}{2}\right) \tag{21}$$

satisfies this functional equation. It was also shown by Gessel and Xin that

$$f(y)^m = \sum_{n=0}^{\infty} \frac{m}{3n+m} \binom{3n+m}{n} y^n. \tag{22}$$

Therefore

$$g(x,i) = x^i f\left(x^3\right)^{i+1} = \tag{23}$$

$$\sum_{n=0}^{\infty} \frac{i+1}{3n+i+1} \binom{3n+i+1}{n} x^{3n+i} \tag{24}$$

The coefficients of this power series are closely related to the probability $w(n,m)$ of finding the random walker at a vertex of the graph at distance m from the root after n steps. Indeed, we have the following

Proposition 3:

$$w(n,m) = \begin{cases} \frac{1}{2n+1}\binom{3n}{n} = \frac{1}{3n+1}\binom{3n+1}{n} & m = 0 \\ \frac{m}{3n+m}\binom{3n+m}{n} & m > 0 \end{cases}, \tag{25}$$

where m is the 'distance' from the root.

Any step from a Type A point will increase the distance of the random walker from the root by 1, a step from type B will either increase the distance by 1, or decrease it by 2 (with 0.5 probability). As established earlier, there are F_m type A points and F_{m-1} type B points at distance m from the root. Therefore the probability of moving one unit away from the root is $\hat{p}(m) = \frac{F_m + \frac{1}{2}F_{m-1}}{F_{m+1}}$ and the probability of decreasing the distance by 2 is $\hat{q}(m) = 1 - p(m) = \frac{\frac{1}{2}F_{m-1}}{F_{m+1}}$. In the large m limit these probabilities are $\hat{p} = \frac{1+\sqrt{5}}{4}$ and $\hat{q} = \frac{1}{3+\sqrt{5}}$, respectively.

Based on this, we will replace the random walk on the cactus by that on the line of integers, where the probabilities of the random walker moving one step to the right and moving two steps to the left are $\hat{p}$ and $\hat{q}$, respectively. Using these probabilities, at every timestep the expected value of the increase in distance is $E = \hat{p} - 2\hat{q} = \frac{\sqrt{5}}{3+\sqrt{5}} \approx 0.427$.

Having calculated the expected increase, we can replace this random walk with a simpler one where the walker only moves one step to the right or left. The expected increase in distance for this walk would be $\hat{q} - (1 - \hat{q}) = 2\hat{q} - 1$. Equating this with the mean increase E, we find that the probability of moving to the right is $p = \frac{3\sqrt{5}-1}{8} \approx 0.7135$ and consequently the probability of moving one step to the left is $q = \frac{9-3\sqrt{5}}{8} \approx 0.2865$.

For a simple one-step biased random walk the mean displacement is $\langle x_N \rangle = \sum_{x=-N}^{N} x P_N(x)$, where $P_N(x)$ is the probability that the walker has a displacement of x after N steps. The mean square displacement $\left\langle x_N^2 \right\rangle = \sum_{x=-N}^{N} x^2 P_N(x)$ is given by

$$\left\langle x_N^2 \right\rangle = \langle x_N \rangle^2 + \left\langle \Delta x_N^2 \right\rangle, \tag{26}$$

where $\left\langle \Delta x_N^2 \right\rangle$ is the variance. Using well-known results for the mean displacement and variance yield

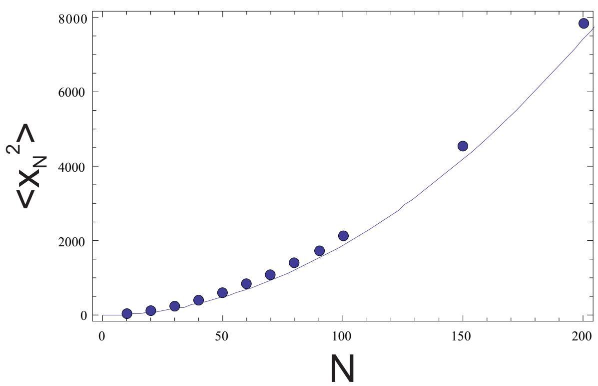
$$\langle x_N \rangle = (p-q)\, N = \frac{3\sqrt{5}-5}{4} N, \tag{27}$$

$$\left\langle \Delta x_N^2 \right\rangle = 4pqN = \frac{3\left(5\sqrt{5}-9\right)}{8} N, \tag{28}$$

and the mean square displacement is therefore

$$\left\langle x_N^2 \right\rangle = \frac{\left(35 - 15\sqrt{5}\right) N^2 + \left(15\sqrt{5} - 27\right) N}{8}. \tag{29}$$

Figure 4 shows the comparison between the mean square displacement of a simulated random walker (10000 runs for each N) on the Husimi cactus and the approximation (29). For smaller values of N some error is expected, however for large N the match is excellent.



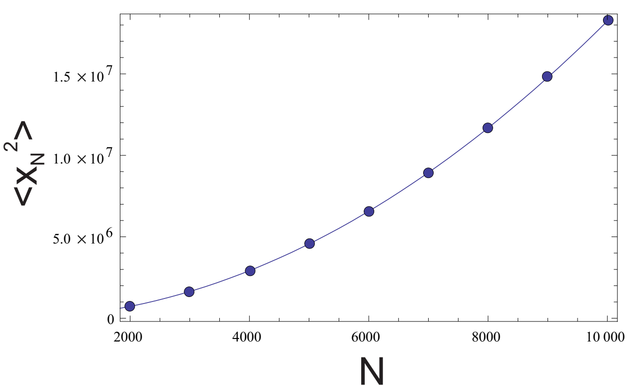


Fig. 4. Comparison between the mean square displacement of a simulated random walker on the Husimi cactus and the approximation (29)

V. ACKNOWLEDGMENTS

The author gratefully acknowledges support from the National Science Fundation CAREER Award CMMI-0846783.

REFERENCES

[1] M. Embree, and L.N. Trefethen, Growth and Decay of Random Fibonacci Sequences. Proceedings: Mathematical, Physical and Engineering Sciences 455, 2471-2485, 1999

[2] T. Kalmár-Nagy, The random Fibonacci recurrence and the visible points of the plane. J. Phys. A: Math. Gen 39, L323-L328, 2006

[3] T. Kalmár-Nagy, Random matrix approach to linear control systems with network-induced delays, IEEE Conference on Decision and Control, pp. 5210–5214, 2007

[4] N. MacDonald, Biological Delay Systems: Linear Stability Theory, Cambridge University Press, New York, 1989

[5] Y. Kuang, Delay Differential Equations with Applications to Population Dynamics, Academic Press, New York, 1993

[6] A. Beuter, J. Bélair, and C. Labrie, Feedback and delays in neurological disease: A modeling study using dynamical systems, Bulletin of Mathematical Biology, 55, pp. 525-541, 1993

[7] D. Pieroux, T. Erneux, T. Luzyanina, K. Engelborghs, Interacting pairs of periodic solutions lead to tori in lasers subject to delayed feedback, Physical Review E, 63(3), p. 36211, 2001

[8] K. Pyragas, Continuous control of chaos by self-controlling feedback, Physical Review Letters, 78(2), pp. 421-428, 1992

[9] H.Y. Hu and Z.H. Wang, Dynamics of Controlled Mechanical Systems with Delayed Feedback, Springer, Berlin, 2002

[10] N. Chopra, M.W. Spong, S. Hirche, M. Buss, Bilateral teleoperation over the internet: the time varying delay problem, American Control Conference, 2003

[11] J. Marescaux, J. Leroy, M. Gagner, F. Rubino, D. Mutter, M. Vix, and S.E. Butner, and M.K. Smith, Transatlantic robot-assisted telesurgery, Nature, 413(6854), pp. 379-80, 2001

[12] P. Ogren, E. Fiorelli, N.E. Leonard, Cooperative control of mobile sensor networks: Adaptive gradient climbing in a distributed environment, IEEE Transactions on Automatic Control, 49(8), pp. 1292–1302, 2004

[13] M.O. Alhalabi, S. Horiguchi, S. Kunifuji, An experimental study on the effects of network delay in cooperative shared haptic virtual environment, Computers & Graphics, 27(2), pp. 205-213, 2003

[14] D. Swaroop and J.K. Hedrick, 1995, String stability of interconnected systems, In Proceedings of the American Control Conference, **3**, pp. 1806-1810, 1995

[15] D. Viswanath, Random Fibonacci Sequences and the Number 1.13198824. Mathematics of Computation, 69, pp. 1131-1155, 2000

[16] R. Krtolica, U. Ozguner, H. Chan, H. Goktas, J. Winkelman, and M. Liubakka, Stability of linear feedback systems with random communication delays, International Journal of Control, 59(4), pp. 925-953, 1994.

[17] Y. Ji, H.J. Chizeck, X. Feng, K.A. Loparo, Stability and control of discrete-time jump linear systems, Control Theory and Advanced Technology, Vol. 7, No. 2, pp. 247-270, 1991

[18] L. Xiao, A. Hassibi, and J. How, Control with random communication delays via a discrete-time jump system approach, In Proceedings of the American Control Conference, 2000

[19] R. Burioni and D. Cassi, Random walks on graphs: ideas, techniques and results, J. Phys. A: Math. General **38**, 45-78, 2005

[20] J. West, Generating trees and forbidden subsequences. Discrete Math 157, pp. 363-374, 1996

[21] I.M. Gessel, and G. Xin, The Generating Function of Ternary Trees and Continued Fractions. the electronic journal of combinatorics 13, 1, 2006.

[22] H.S. Wilf, Generatingfunctionology, Academic Press Boston, 1990.