Gain-Scheduled $H_2$ Filter Synthesis via Polynomially Parameter-Dependent Lyapunov Functions with Inexact Scheduling Parameters

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Abstract—This paper addresses the design problem of Gain-Scheduled (GS) $H_2$ filters for Linear Parameter-Varying (LPV) systems under the condition that only inexact measured scheduling parameters are available. The state-space matrices of the LPV systems are supposed to be polynomially parameter dependent and those of filters which are to be designed are supposed to be rationally parameter dependent. The uncertainties in the measured scheduling parameters are supposed to lie in a priori defined convex set. Using structured polynomially Parameter-Dependent Lyapunov Functions (PDLFs), we give a design method of GS $H_2$ filters, which are robust against the uncertainties in the measured scheduling parameters, in terms of parametrically affine Linear Matrix Inequalities (LMIs). Our proposed method includes robust filter design as a special case. A numerical example demonstrates the effectiveness of our method.

I. INTRODUCTION

After the proposition of the filter design using Linear Matrix Inequalities (LMIs) in [1], many papers on filter design problem have been reported, e.g. [2]–[12]. Some of them tackle robust filter design problem for Linear Parameter-Varying (LPV) systems or Linear Time-Invariant Parameter-Dependent (LTIPD) systems, and others tackle Gain-Scheduled (GS) filter design problem for LPV systems. Generally speaking, designing filters for LPV/LTIPD systems needs to solve Parameter-Dependent LMIs (PDLMs). On this issue, several powerful methods, such as, Sum-of-Squares (SOS) approach [13]–[15], matrix dilation or Slack Variable (SV) approach [16], coefficient check approach [17], have been proposed and their effectiveness has also been demonstrated. Thus, in a sense, we can easily design filters for LPV/LTIPD systems with the aid of those methods.

As illustrated in [4], [8], [11], [12], if the scheduling parameters which describe the changes of the plant dynamics are available, it is well known that GS filters have better performance than robust filters. However, existing design methods of GS filters assume that the scheduling parameters are exactly measurable and available, which is impossible in real systems. Thus, design methods of GS filters which are robust against the uncertainties in the measured scheduling parameters have been desired. Several papers have already tackled similar problems [18], [19]; however, to author’s knowledge, very few results on the design problem for GS filters which exploit inexact scheduling parameters have been reported. In [20], a design method on GS $H_\infty$ filters which exploit inexact measured scheduling parameters has been successfully proposed. In this note, we show the counterpart result for $H_2$ filter design problem1. Similarly to [20], the uncertainties are supposed to be in a priori defined convex set, which can include a hyper-rectangle, and to vary with time. Using structured polynomially Parameter-Dependent Lyapunov Functions (PDLFs), we propose a design method for GS $H_2$ filters which are robust against the uncertainties in the measured scheduling parameters. We also show that our method encompasses a design method for robust $H_2$ filters as a special case.

This paper is organized as follows: In section II, we briefly review the conventional design method of GS $H_2$ filters using Parameter-Dependent Lyapunov Functions (PDLFs), then define our addressed problem. In section III, we show our design method and give some remarks on our method. In section IV, a simple numerical example is introduced to illustrate our results. Finally, we give concluding remarks.

In this note, we use the following notations. $\langle X \rangle$ is the shorthand notation of $X + X^T$, $0_{n,m}$, $I_n$, and $0$ respectively denote an $n \times m$-dimensional zero matrix, an $n$-dimensional identity matrix and an appropriately dimensional zero matrix, $R^{n \times m}$ and $S^n$ respectively denote sets of $n \times m$-dimensional real matrices and $n \times n$-dimensional symmetric real matrices, $\otimes$ denotes Kronecker product, and $*$ in matrices denotes an abbreviated off-diagonal block. For a symmetric matrix $X = \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ \vdots & \ddots & \vdots \\ X_{m1} & \cdots & X_{mm} \end{bmatrix} \in S^{nm}$, in which the dimensions of $X_{ij}$ are $n \times n$, $\text{Tr}_n(X)$ denotes $\begin{bmatrix} \text{Tr}(X_{11}) & \cdots & \text{Tr}(X_{1m}) \\ \vdots & \ddots & \vdots \\ \text{Tr}(X_{m1}) & \cdots & \text{Tr}(X_{mm}) \end{bmatrix}$.

II. PRELIMINARIES

In this section, we first review the conventional design method of GS $H_2$ filters using PDLFs, in which change-of-variables [21] is applied, then define our addressed problem.

1Rigorously speaking, $H_2$ performance cannot be defined in our addressed problem, because as the plant is an LPV system the augmented system with the plant and designed GS filter is also an LPV system. We use this terminology so that readers can easily grasp our addressed problem with admitting that this terminology is slightly abused.
A. Conventional Design Method

In this subsection, we briefly review the conventional design method of GS $H_2$ filters for LPV systems using PDLFs.

Suppose that a stable LPV plant system $G(\theta)$ with $k$ independent scalar parameters $\theta = [\theta_1 \cdots \theta_k]^T$ is given.

$$G(\theta) : \begin{cases} \dot{x} = A(\theta)x + B(\theta)w \\ z = C(\theta)x + D(\theta)w , \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector with $x = 0$ at $t = 0$, $w \in \mathbb{R}^{n_w}$ is the disturbance input vector, $z \in \mathbb{R}^n$ is the vector of signals to be estimated, and $y \in \mathbb{R}^m$ is the vector of measurement outputs. The parameters $\theta_i$, which represent plant uncertainties or the changes of plant dynamics, are supposed to be time-varying (possibly time-invariant).

The ranges of $\theta_i$ and $\hat{\theta}_i$ (the derivative of $\theta_i$ with respect to time) are assumed to lie in closed convex sets $\Omega_\theta$ and $\Lambda_\theta$, which are known in advance: $\theta(t) \in \Omega_\theta$, $\hat{\theta}(t) \in \Lambda_\theta$, $\forall t \geq 0$, where $\hat{\theta} = [\hat{\theta}_1 \cdots \hat{\theta}_k]^T$.

For LPV system (1), we consider a full-order GS filter $F(\theta)$.

$$F(\theta) : \begin{cases} \dot{x}_f = A_f(\theta)x_f + B_f(\theta)y \\ \dot{z} = C_f(\theta)x_f , \end{cases} \tag{2}$$

where $x_f \in \mathbb{R}^n$ is the state vector with $x_f = 0$ at $t = 0$, $z \in \mathbb{R}^n$ is the vector of estimated signals of $z$ (see Fig. 1).

The state-space matrices $A_f(\theta), B_f(\theta)$ and $C_f(\theta)$, which have appropriate dimensions, are to be designed.

Similarly to [21], the following lemma is easily derived from Lemma 4 in the appendix, which is for the design of GS $H_2$ output-feedback controllers, with a candidate of PDLFs being set as $x_a^T X_a(\theta)^{-1} x_a$ where $x_a = [x^T \ x_f^T]^T$.

**Lemma 1:** For a given positive number $\gamma_2$, suppose that there exist continuously differentiable positive definite matrices $R(\theta) \in \mathbb{S}^n$ and $S(\theta) \in \mathbb{S}^n$, a positive definite matrix $N(\theta) \in \mathbb{S}^{n_w}$, and matrices $A_f(\theta) \in \mathbb{R}^{n \times n}$, $B_f(\theta) \in \mathbb{R}^{n \times n_u}$ and $C_f(\theta) \in \mathbb{R}^{n \times n}$ such that (3), (4) and (5) hold. Then, a stable GS filter, whose state-space matrices are given as in (7), satisfies (6) for $w = \delta(0) w_0$, where $w_0$ is a random variable satisfying $E\{w_0 w_0^T\} = I_{n_w}$ and $\delta(\cdot)$ is Dirac’s delta function.

$$\begin{bmatrix} N(\theta) & B(\theta) \\ S(\theta) B(\theta) + B_f(\theta) D(\theta) \end{bmatrix} \begin{bmatrix} R(\theta) & I_n \\ I_n & S(\theta) \end{bmatrix}^{*} \begin{bmatrix} I_n \\ S(\theta) \end{bmatrix} > 0, \quad \forall \theta \in \Omega_\theta \tag{3}$$

**Remark 1:** Note that the online calculation for the state-space matrices of GS filters designed by Lemma 2 is much simpler than that designed by Lemma 1. Furthermore, GS filters designed by Lemma 2 need no derivatives of parameters in their state-space matrices even if PDLFs are used. On the other hand, Lemma 2 is as little conservative as Lemma 1.
As shown above, we can design GS $H_2$ filters with ease. However, these methods assume that the scheduling parameters are exactly measurable and available, which is almost impossible in real systems. In other words, the designed filters cannot assure a priori defined performance when they are implemented to real world.

B. Problem Definition

In this subsection, we define our addressed problem.

Suppose that the state-space matrices of LPV system (1) are given as follows.

$$A(\theta) = A \left( \theta \otimes I_n \right), \quad \hat{A} \in \mathbb{R}^{n \times n \sigma},$$
$$B(\theta) = B \left( \theta \otimes I_{n_w} \right), \quad \hat{B} \in \mathbb{R}^{n \times n_w \sigma},$$
$$C(\theta) = C \left( \theta \otimes I_n \right), \quad \hat{C} \in \mathbb{R}^{n_\sigma \times n \sigma},$$
$$D(\theta) = D \left( \theta \otimes I_{n_w} \right), \quad \hat{D} \in \mathbb{R}^{n_\sigma \times n_{w\sigma}},$$

where

$$\theta_i = [\theta_i^1, \theta_i^2, \ldots, \theta_i^{m_i}]^T \in \mathbb{R}^{\sigma_i}, \quad \hat{\theta} = [\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_k]^T \in \mathbb{R}^\sigma.$$ 

Here, $\sigma_i = m_i + 1$ and $\sigma = \sigma(1, k) = \prod_{i=1}^k \sigma_i$. Both of $\sigma(k + 1, k)$ and $\sigma(1, 0)$ are defined as 1. The matrices $A$, $B$, etc. are row-block matrices, each of which represents the coefficient associated with one monomial in $\theta$.

The ranges of $\theta_i$ and $\theta_i^j$ are assumed to lie in closed convex sets $\Omega_\theta$ and $\Lambda_\theta$: $\theta(t) \in \Omega_\theta$, $\theta(t) \in \Lambda_\theta$, $\forall t \geq 0$. The vertex sets of $\Omega_\theta$ and $\Lambda_\theta$ are respectively given as $\text{ver}(\Omega_\theta)$ and $\text{ver}(\Lambda_\theta)$.

Now we define GS $H_2$ filters which exploit inexactly measured scheduling parameters. It is supposed that the scheduling parameters $\theta_i$ are not exactly measured, but measured with some uncertainties $\delta_i$. That is, the $i$-th scheduling parameter $\theta_i$ is assumed to be measured as $\theta_i + \delta_i$.

The range of $\delta_i$ and its estimated range of $\delta_i$ (the derivative of $\delta_i$ with respect to time) are assumed to lie in closed convex sets $\Omega_\delta$ and $\Lambda_\delta$, which are known in advance:

$$\delta(t) \in \Omega_\delta, \quad \forall t \geq 0, \quad \delta = [\delta_1, \ldots, \delta_k]^T,$$

where $\delta_i \in \Omega_\delta$ and $\delta_i \in \Lambda_\delta$. The vertex sets of $\Omega_\delta$ and $\Lambda_\delta$ are respectively given as $\text{ver}(\Omega_\delta)$ and $\text{ver}(\Lambda_\delta)$.

We now define the filter to be designed as follows.

$$F(\theta, \delta) : \left\{ \begin{array}{ll}
\dot{x}_f &= A_f(\theta, \delta)x_f + B_f(\theta, \delta)y, \\
\dot{\hat{z}} &= C_f(\theta, \delta)x_f
\end{array} \right.,$$

where $x_f \in \mathbb{R}^n$ is the state vector with $x_f = 0$ at $t = 0$, and $\hat{z} \in \mathbb{R}^{n_w}$ is the vector of estimated signals of $z$ (see Fig. 2).

The state-space matrices in (14), which have appropriate dimensions, are to be designed.

If some parameters are assumed to be unmeasurable, i.e., the parameters represent the plant uncertainties, then the state-space matrices of $F(\theta, \delta)$ should be independent of such parameters. Similarly, if a robust filter is to be designed, the state-space matrices in (14) should be all constant. On these issues, we give remarks after showing our design method.

The augmented system $G_a(\theta, \delta)$ comprising $G(\theta)$ and $F(\theta, \delta)$, which is depicted in Fig. 2, is given as follows.

$$G_a(\theta, \delta) : \left\{ \begin{array}{ll}
\dot{x}_a &= A_a(\theta, \delta)x_a + B_a(\theta, \delta)w, \\
\dot{\hat{z}} &= C_a(\theta, \delta)x_a
\end{array} \right.,$$

where

$$A_a(\theta, \delta) = A(\theta) + 0_{n, n}, \quad B_a(\theta, \delta) = \begin{bmatrix}
B_f(\theta, \delta) & C_f(\theta, \delta)
\end{bmatrix},$$
$$C_a(\theta, \delta) = \begin{bmatrix}
C(\theta) & -C_f(\theta, \delta)
\end{bmatrix}.$$
exponentially stable for all pairs \( (\theta, \hat{\theta}, \delta, \hat{\delta}) \in \Omega_\theta \times \Lambda_\theta \times \Omega_\delta \times \Lambda_\delta \) and satisfies (16).

\[
\begin{align*}
N(\theta, \delta) &> 0, \quad \forall (\theta, \delta) \in \Omega_\theta \times \Omega_\delta \\
\left[ (P_n(\theta, \delta)A_2(\theta, \delta)) + \frac{dP_n(\theta, \delta)}{dt} \right]C_n(\theta, \delta) &- \dot{I}_n \leq 0, \quad \forall (\theta, \hat{\theta}, \delta, \hat{\delta}) \in \Omega_\theta \times \Lambda_\theta \times \Omega_\delta \times \Lambda_\delta \\
\gamma_2^2 - \text{Tr} \left( N(\theta, \delta) \right) &> 0, \quad \forall (\theta, \delta) \in \Omega_\theta \times \Omega_\delta
\end{align*}
\]  

(17) (18) (19)

In the next section, we apply Lemma 3 for solving Problem 1.

III. MAIN RESULTS

In this section, we first show our proposed method for Problem 1. Then, we show some extensions of the method in cases that rate bounds for some parameters cannot be estimated and some parameters are not available.

A. Proposed Method

Considering Remark 1, we use PDLFs which are structured similarly to (9).

Before showing our method, we give several definitions.

\[
e = [1 \ 0_{1,\sigma-1}]^T \in \mathbb{R}^\sigma \\
\eta_{i}[\infty] = \begin{bmatrix} I_{m_i} \\ 0_{1,m_i} \end{bmatrix}, \quad \eta_{i}[0] = -\begin{bmatrix} 0_{1,m_i} \\ I_{m_i} \end{bmatrix},
\]

\[
\eta_i(\theta) = \theta \eta_{i}[\infty] + \eta_{i}[0], \\
\Psi_i[\infty] = I_{\sigma(i,1-i)} \otimes \eta_{i}[\infty] \otimes I_{\sigma(i+1,k)}, \\
\Psi_i[0] = I_{\sigma(i,1-i)} \otimes \eta_{i}[0] \otimes I_{\sigma(i+1,k)}, \\
\Psi_i(\theta) = \theta \Psi_{i}[\infty] + \Psi_{i}[0].
\]

Here, \( \sigma_i \) denotes \( \sigma m_i / \sigma_1 \). Note that \( \bar{\theta}^T \Psi_i(\theta_i) = 0 \) holds.

A candidate of PDLFs is set as \( x_a^T P_S(\theta, \delta)x_a \) with the following \( P_S(\theta, \delta) \).

\[
P_S(\theta, \delta) = \begin{bmatrix} P(\theta, \delta) S(\theta, \delta) \\ S(\theta, \delta)^T S(\theta, \delta) \end{bmatrix},
\]

(20)

where \( P(\theta, \delta) \in S^m \) and \( S(\theta, \delta) \in S^m \).

The parameter-dependency of \( P_S(\theta, \delta) \) is set in the sequel. Matrix \( P(\theta, \delta) \) is set as follows.

\[
P(\theta, \delta) = \begin{bmatrix} \bar{\theta} \otimes I_n^T \\ \dot{P} + e \otimes P(\delta) \end{bmatrix},
\]

(21)

where \( \bar{\theta} \otimes \bar{I}_n^T \) is a block column matrix composed of \( n \times n \)-dimensional symmetric matrices and \( \bar{P}_\delta \in S^m \). Noting that \( \bar{\theta}^T e = 1, P(\theta, \delta) \) is expressed as

\[
\frac{1}{2} \left( \bar{\theta} \otimes I_n^T \right) \left( \dot{P}(e^T \otimes I_n) + (e e^T) \otimes P(\delta) \right) \left( \bar{\theta} \otimes I_n \right).
\]

Matrix \( S(\theta, \delta) \) is set as follows.

\[
S(\theta, \delta) = S_0 + \sum_{i=1}^k (\theta_i + \delta_i) S_i,
\]

(22)

where \( S_i \in S^n \) \( (i = 0, \ldots, k) \). That is, \( S(\theta, \delta) \) affinely depends on the inexactely measured scheduling parameters.

As PDLFs are considered, \( \frac{dP_n(\theta, \delta)}{dt} \) appears in (18). Accordingly to [20], the term \( \frac{dP_n(\theta, \delta)}{dt} \) is expressed as

\[
\left( \bar{\theta} \otimes I_n \right)^T \frac{1}{2} \sum_{i=1}^k \left[ Y_i \left( \bar{\Xi}_i(\theta_i), \bar{P}, \bar{\delta}_i P_\delta \right) \right] \left( \bar{\theta} \otimes I_n \right),
\]

(23)

where \( \bar{Y}_i \left( \Xi_i(\theta_i), \bar{P}, \bar{\delta}_i P_\delta \right) \) is defined as

\[
\left( I_{\sigma(i,1-i)} \otimes \Xi_i(\theta_i) \otimes I_{\sigma(i+1,k)} \right)^T \bar{P}(e^T \otimes I_n)
\]

\[
+ (e e^T) \otimes (\bar{\delta_i} P_\delta).
\]

with

\[
\Xi_i(\theta_i) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}
\]

Under these preliminaries, the following theorem is obtained.

Theorem 1: For a given positive number \( \gamma_2 \), suppose that there exist a block column matrix \( \bar{P} \) which is composed of \( n \times n \)-dimensional symmetric matrices, parametrically affine symmetric matrices \( P(\delta) \) and \( S(\theta, \delta) \), which are respectively defined as in (21) and (22), and \( \bar{N}(\delta) \in S^{n \times \sigma}, \) and matrices \( A_f \in \mathbb{R}^{n \times n}, B_f \in \mathbb{R}^{n \times n}, C_f \in \mathbb{R}^{n \times n}, \) \( F_i \in \mathbb{R}^{n \times n}, M_i \in \mathbb{R}^{n \times n}, \) and \( H_i \in \mathbb{R}^{n \times n} \) such that (24), (25), and (26), the first two of which are at the top of the next page, hold. Then, filter \( \bar{F}(\theta, \delta) \), whose state-space matrices are given as in (27), is stable and satisfies (16).

\[
\gamma_2^2 (e e^T) - \text{Tr}_{\mathbb{R}^{n \times n}} \left( \bar{N}(\delta) + (\Psi_i(\theta_i) H_i) \right) > 0,
\]

(26)

\[
\begin{bmatrix} [A_f(\theta, \delta) B_f(\theta, \delta)] \end{bmatrix} = \begin{bmatrix} S(\theta, \delta)^{-1} & 0 & -I_n \\ 0 & 1 & I_n \\ -I_n & -1 & 0 \end{bmatrix} \begin{bmatrix} A_f(\theta, \delta) B_f(\theta, \delta) \end{bmatrix},
\]

(27)

In (24) and (25), \( \Gamma \) is defined as

\[
\Gamma = \begin{bmatrix} \bar{P} + e \otimes P(\delta) \\ S(\theta, \delta) \bar{A} + B_f(\theta, \delta) \bar{C}_2 \end{bmatrix},
\]

\[
\begin{bmatrix} A_f(\theta, \delta) B_f(\theta, \delta) \end{bmatrix},
\]

and matrices

\[
\begin{bmatrix} [A_f(\theta, \delta) B_f(\theta, \delta)] \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

\[
\begin{bmatrix} [A_f(\theta, \delta) B_f(\theta, \delta)] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

\[
\text{diag} \left( \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)
\]

Proof: Note that all the inequalities (24), (25), and (26) are affine with respect to \( \theta \) and/or \( \delta \). Thus, if they hold at all the vertices of the related parameters, then they hold for all the related parameters.
to (24) from the left and the right respectively lead to (17) with $N(\theta, \delta)$ and $P_n(\theta, \delta)$ being respectively set as $\left(\hat{\theta} \otimes I_n\right)^T \hat{N}(\delta) \left(\hat{\theta} \otimes I_n\right)$ and $P_S(\theta, \delta)$ in (20) after conducting the change-of-variable $S(\theta, \delta)B_f(\theta, \delta) = B_f(\theta, \delta)$. Similarly, pre- and post multiplications of $\left[\hat{\theta} \otimes I_n \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ I_n \ 0 \ 0\right]^T$ and its transpose to (25) from the left and the right respectively lead to (18) after conducting the change-of-variables $S(\theta, \delta)A_f(\theta, \delta) = A_f(\theta, \delta)$ and $C_f(\theta, \delta) = C_f(\theta, \delta)$. Similar algebraic manipulations to (26) lead to (19). This completes the proof.

B. Parameter Setting for Decision Matrices

In the sequel, we give several remarks on the parameter setting for decision matrices in Theorem 1.

We first give remarks with respect to the bounds for $\hat{\theta}_i$ and $\hat{\delta}_i$ under the assumption that the $i$-th scheduling parameter is available. We next give a remark on the availability of the scheduling parameters.

1) Rate Bounds for Parameters: Theorem 1 assumes that the rate bound for $\delta_i$ can be estimated. However, it may be very difficult to estimate it. In such case, the rate bound should be set as infinite, i.e., $|\delta_i| = \infty$. Consequently, the matrices $P_{\delta_i}$ and $S_i$ are set zeros. Then, both $P(\theta, \delta)$ and $S(\theta, \delta)$ are independent of $\delta_i$, which removes the derivatives of $\delta_i$ in $P_{\delta_i}(\theta, \delta)$ from (25). However, this modification introduces some conservatism due to the restriction of parameter-dependency for $P(\theta, \delta)$ and $S(\theta, \delta)$. Especially, $S(\theta, \delta)$ is set as independent of the $i$-th scheduling parameters.

Similarly, if the bound for $\hat{\theta}_i$ cannot be estimated, both $P(\theta, \delta)$ and $S(\theta, \delta)$ are set to be independent of $\theta_i$ and $\delta_i$. This modification removes the derivatives of $\theta_i$ and $\delta_i$ in $P_{\theta_i}(\theta, \delta)$ from (25). In this case, the parameter $\theta_i$ can move arbitrarily fast and the state-space matrices in (27) become parametrically affine with respect to the $i$-th parameter. Furthermore, if all the matrices $S_i$ ($i = 1, \ldots, k$) are set zeros, then $S(\theta, \delta)$ becomes constant, which reduces online numerical complexity for obtaining the state-space matrices of $F(\theta, \delta)$.

We summarize the above discussions in Table I.

2) Availability of Parameters: In Theorem 1, all the scheduling parameters are assumed to be available with some uncertainties. However, it may happen that some parameters cannot be available, e.g. the parameter represents the uncertainties of the plant. In such case, if the $i$-th scheduling parameter cannot be available, then the related matrices in $A_f$, $B_f$, and $S(\theta, \delta)$ are set to be zeros. Consequently, Theorem 1 produces GS filters which are independent of the $i$-th scheduling parameter.

Similarly, if all the parameters cannot be available, then $P_{\theta}(\theta, \delta)$ is set as $P(\theta) = \begin{bmatrix} P_0 \ S \ S \\ S \ S \ S \end{bmatrix}$, where $P(\theta)$ has the same definition as $P(\theta, \delta)$ in (21) but all the matrices $P_{\delta_i}$ ($i = 1, \ldots, k$) being set as zeros. This can be set without loss of generality, because the augmented system $G_a(\theta, \delta)$ no longer depends on $\delta$. Similarly, matrix $N(\delta)$ can be set as constant.

Then, the following corollary on robust filter design is derived from Theorem 1.

Corollary 1: For a given positive number $\gamma_2$, suppose that there exist a block column matrix $P$, constant symmetric matrices $S_0$ and $N_0$, and matrices $A_f \in \mathbb{R}^{n \times n}$, $B_f \in \mathbb{R}^{n \times n}$, $C_f \in \mathbb{R}^{n \times n}$, $F_i$, $M_i$, and $H_i$ such that (24), (25), and (26) in which the followings are set:

$P(\delta) = 0_{n,n}$, $S(\theta, \delta) = S_0$, $N(\delta) = N_0$, $\delta = 0$, $A_f(\theta, \delta) = A_f$, $B_f(\theta, \delta) = B_f$, $C_f(\theta, \delta) = C_f$ hold. Then, filter $F(\theta, \delta)$ whose state-space matrices are given as

\[
\begin{bmatrix}
A_f & B_f \\
C_f & I
\end{bmatrix}^{-1}
\begin{bmatrix}
0 & 1 \\
0 & I_n
\end{bmatrix}
\begin{bmatrix}
A_f & B_f \\
C_f & I
\end{bmatrix}
\]

satisfies (16).

IV. Numerical Examples

We consider Example 2 in [11]. The state-space matrices in (1) is given as
To use robust scheduling parameters heavily affect the online numerical complexity. Where the scheduling parameter vector is \( \theta \) and \( \xi \) is the uncertainty in the scheduling parameters. Thus, if the uncertainties are so large, it is better to use robust \( H_2 \) filters instead of GS \( H_2 \) filters in terms of the online numerical complexity.

For reference, we also design robust \( H_2 \) filters using corollary 1. The results are shown in Table III.

To confirm that the designed GS \( H_2 \) filters have robustness against the uncertainties in the measured scheduling parameters, we check the maximum \( H_2 \) performance for the augmented system \( G_a(\theta, \delta) \) with the scheduling parameters for filters being set as \( \theta_i \pm \xi \) using the real scheduling parameters \( \theta_i \), while the real scheduling parameter is set as \( \theta_i = \pm 1 \) (i = 1, 2). Thus, totally, 16 combinations, i.e. 4 combinations for \( \theta_i \) and 4 combinations for \( \delta_i \), are considered. The result is shown in Table IV, in which the values of \( \zeta \) and \( \xi \) denote their values when designing GS filters. It is confirmed that the designed GS \( H_2 \) filters have robustness against the supposed uncertainties in the measured scheduling parameters at least when the scheduling parameters and their uncertainties are both frozen, i.e. \( \bar{\theta}_i = \bar{\delta}_i = 0 \).

Similarly, a posteriori check for robust filters are conducted and the results are shown in Table V.

### V. CONCLUSIONS

This note tackles the design problem of Gain-Scheduled (GS) \( H_2 \) filters for Linear Parameter-Varying (LPV) systems whose state-space matrices are polynomially parameter-dependent. Considering that it is almost impossible to obtain the exact values of the scheduling parameters in real systems, it is supposed that the scheduling parameters are measured with some uncertainties which are a priori defined. For this practical problem, we give a formulation for GS \( H_2 \) filters exploiting inexact scheduling parameters using polynomially Parameter-Dependent Lyapunov Functions (PDLFs) in terms of parametrically affine Linear Matrix Inequalities (LMIs). A numerical example borrowed from the literature supports our results.

### APPENDIX

In the sequel, we show a design method of GS \( H_2 \) controllers for LPV systems using the method in [21]

Consider the following LPV system

\[
\dot{x} = A(\theta)x + B_1(\theta)w + B_2(\theta)u
\]

\[
g(\theta) : \begin{aligned}
z &= C_1(\theta)x + D_{12}(\theta)u \\
y &= C_2(\theta)x + D_{21}(\theta)w
\end{aligned}
\]

where \( x \in \mathbb{R}^n \) is the state vector with \( x = 0 \) at \( t = 0 \), \( w \in \mathbb{R}^{n_w} \) is the disturbance input vector, \( u \in \mathbb{R}^{n_u} \) is the control input vector, \( z \in \mathbb{R}^{n_z} \) is the performance output vector, and \( y \in \mathbb{R}^{n_y} \) is the measurement output vector. The parameter vector \( \theta = [\theta_1 \cdots \theta_k]^T \), which denotes the changes of the plant dynamics, are time-varying (possibly time-invariant). The ranges of \( \theta \) and the derivative of \( \theta \) are assumed to lie in closed convex sets: \( \theta(t) \in \Omega_\theta, \dot{\theta}(t) \in \Delta_\theta, \forall t \geq 0 \), where \( \theta = [\theta_1 \cdots \theta_k]^T \).

Consider the following full-order GS controller

\[
\begin{aligned}
\dot{\theta}_c &= A_c(\theta)x_c + B_c(\theta)y \\
u &= C_c(\theta)x_c
\end{aligned}
\]
where \( x_c \in \mathbb{R}^n \) denotes the state vector with \( x_c = 0 \) at \( t = 0 \).

Then, the closed-loop system is given as follows.

\[
G_{cl}(\theta) : \begin{cases}
\dot{x}_{cl} = A_{cl}(\theta)x_{cl} + B_{cl}(\theta)w \\
z = C_{cl}(\theta)x_{cl}
\end{cases},
\]

where \( x_{cl} = [x^T \, x_c^T]^T \), and

\[
A_{cl}(\theta) = \begin{bmatrix} A(\theta) & B_2(\theta)C_c(\theta) \\
B_1(\theta)C_2(\theta) & A_c(\theta) \end{bmatrix},
B_{cl}(\theta) = \begin{bmatrix} B_1(\theta) \\
B_1(\theta)D_{21}(\theta) \end{bmatrix},
C_{cl}(\theta) = \begin{bmatrix} C_1(\theta) D_{12}(\theta)C_c(\theta) \end{bmatrix}.
\]

Then, the following lemma is obtained from the result in [22].

**Lemma 4:** For a given positive integer \( \gamma_2 \), if there exist continuously differentiable positive definite matrices \( R(\theta) \in S^m \) and \( S(\theta) \in S^n \), a positive definite matrix \( N(\theta) \in S_{\text{pos}}^w \), and matrices \( A_c(\theta) \in \mathbb{R}^{n \times m} \), \( B_c(\theta) \in \mathbb{R}^{m \times n} \) and \( C_c(\theta) \in \mathbb{R}^{m \times n} \) such that (31), (32), which is at the top of the next page, and (33) hold, then GS controller, whose state-space matrices are given as in (35), makes the closed-loop system composed of (28) and (29) exponentially stable for all pairs \((\theta, \hat{\theta}) \in \Omega \times \Lambda_0\) and satisfies (34).

\[
\begin{cases}
N(\theta) * \\
B_1(\theta) \\
S(\theta)B_1(\theta) + B_c(\theta)D_{21}(\theta)
\end{cases}
\begin{bmatrix}
R(\theta) & I \\
I & S(\theta)
\end{bmatrix}
> 0,
\forall \theta \in \Omega_0
\tag{31}
\]

\[
\gamma_2^2 - \text{Tr} \{N(\theta)\} > 0, \quad \forall \theta \in \Omega_0
\tag{33}
\]

\[
\sup_{(\theta, \dot{\theta}) \in \Omega_0 \times \Lambda_0} E \left( \int_0^{\infty} z^T z \, dt \right) < \gamma_2^2
\tag{34}
\]

\[
\begin{cases}
A_c(\theta) = N(\theta)^{-1} \left[ A_c(\theta) - S(\theta)A(\theta)R(\theta) - B_c(\theta)C_2(\theta)R(\theta) - S(\theta)B_2(\theta)C_c(\theta) \right] + S(\theta)\dot{R}(\theta) + N(\theta)M(\theta)^T M(\theta)^{-T}, \\
B_c(\theta) = N(\theta)^{-1} \dot{B}_c(\theta), \\
C_c(\theta) = \dot{C}_c(\theta)M(\theta)^{-T}
\end{cases}
\tag{35}
\]

where matrices \( M(\theta) \), \( N(\theta) \in \mathbb{R}^{m \times n} \) are arbitrary matrices satisfying \( I_m - R(\theta)S(\theta) = M(\theta)N(\theta)^T \).

Using the following change-of-variables

\[
\dot{A}_c(\theta) = [S(\theta)A(\theta) + N(\theta)B_c(\theta)C_2(\theta)] R(\theta) + [S(\theta)B_2(\theta)C_c(\theta) + N(\theta)A_c(\theta)] M(\theta)^T - S(\theta)\dot{R}(\theta) - N(\theta)\dot{M}(\theta)^T,
\]

\[
\dot{B}_c(\theta) = N(\theta)B_c(\theta),
\dot{C}_c(\theta) = C_c(\theta)M(\theta)^T,
\]

the lemma is easily proved, similarly to [21]. Thus, the proof is omitted.
\[
\begin{bmatrix}
A(\theta)R(\theta) + B_2(\theta)C_c(\theta) - R(\theta) & A(\theta) \\
A_c(\theta) & S(\theta)A(\theta) + B_c(\theta)C_2(\theta) + ˙S(\theta) \\
C_1(\theta)R(\theta) + D_{12}(\theta)C_c(\theta) & C_1(\theta)
\end{bmatrix}^* < 0, \quad \forall (\theta, ˙\theta) \in \Omega_{\theta} \times \Lambda_{\theta}
\]