Anisotropy-Based Bounded Real Lemma

Alexander P. Kurdyukov, Eugene A. Maximov, Michael M. Tchaikovsky

Abstract— **This paper extends the Bounded Real Lemma of the $\mathcal{H}_\infty$-control theory to stochastic systems under random disturbances with imprecisely known probability distributions. The statistical uncertainty is measured in entropy theoretic terms using the mean anisotropy functional. The disturbance attenuation capabilities of the system are quantified by the anisotropic norm which is a stochastic counterpart of the $\mathcal{H}_\infty$-norm. We develop a state-space criterion for the anisotropic norm of a linear discrete time invariant system to be bounded by a given threshold value. The resulting Anisotropy-based Bounded Real Lemma involves an inequality on the determinant of a matrix associated with a parameter-dependent algebraic Riccati equation.**

I. INTRODUCTION

The main concept of the anisotropy-based approach to robust stochastic control, originated in the mid 1990's in [8], [9], [10], is the *anisotropic norm* of systems which builds on the *anisotropy* of random signals. Enhancing the qualitative meaning of this term as it is used in Physics, the anisotropy functional, considered here, is an entropy theoretic measure of the deviation of a probability distribution in Euclidean space from Gaussian distributions with zero mean and scalar covariance matrices. The isotropic Gaussian distributions (whose importance was first recognised in linear regression analysis for justifying the least squares method) are customary models of noise in Linear Quadratic Gaussian (LQG) filtering and control.

The transition from finite-dimensional random vectors to stationary random sequences is carried out in a standard way by defining the *mean anisotropy* of such a sequence as the anisotropy production rate per time step for long segments of the sequence. In application to random disturbances, the mean anisotropy describes the amount of statistical uncertainty which is understood as the discrepancy between the imprecisely known actual noise distribution and the family of nominal models which consider the disturbance to be a Gaussian white noise sequence with a scalar covariance matrix.

The *a-anisotropic norm* quantifies the disturbance attenuation capabilities of a linear discrete time invariant (LDTI) system by the largest ratio of the power norm of the system output to that of the input, provided that the mean anisotropy of the input disturbance does not exceed a given nonnegative parameter a.

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A.P.Kurdyukov and M.M.Tchaikovsky are with the Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia, akurd@ipu.ru, mmtchaikovsky@hotmail.com.

E.A.Maximov is with the Bauman Moscow State Technical University, Moscow, Russia, eugene.maximov@gmail.com.

In the context of robust stochastic control design, which is concerned with suppressing the potentially harmful effects of statistical uncertainty, the anisotropy-based approach offers an important alternative to those control design procedures that are "fine-tuned" for a specific probability law of the disturbance.

As a performance criterion, the minimization of the anisotropic norm of the closed-loop system provides controllers that are less conservative than the $\mathcal{H}_\infty$-controllers and more efficient for correlated disturbance attenuation than the LQG controllers. A state-space solution to the optimal control problem by the minimum anisotropic norm criterion was obtained in [11]. This control design procedure, which yields an internally stabilizing dynamic feedback controller, involves the solution of three cross-coupled algebraic Riccati equations, an algebraic Lyapunov equation and a *mean anisotropy equation* on the determinant of a related matrix.

A natural extension of this approach is the suboptimal anisotropic controller design. The requirement to internally stabilize the closed-loop system still applies. However, instead of minimizing the anisotropic norm of the system, a suboptimal controller is only required to keep it below a given threshold value. Rather than singling out a unique controller, the suboptimal design yields a family of controllers, thus providing freedom to impose additional specifications on the closed-loop system. One of such specifications, for example, is a particular pole placement to achieve desirable transient characteristics.

The suboptimal anisotropic control design requires a state-space criterion for verifying whether the anisotropic norm of a system does not exceed a given value. The well-known Bounded Real Lemma of the $\mathcal{H}_\infty$-control theory relates an upper bound on the $\mathcal{H}_\infty$-norm with the solvability of an algebraic Riccati equation associated with the state-space realization of the system. In extending this result, the present paper develops an Anisotropy-based Bounded Real Lemma (ABRL) for the anisotropic norm as a stochastic counterpart of the $\mathcal{H}_\infty$-norm for LDTI systems under statistically uncertain stationary Gaussian random disturbances with limited mean anisotropy. The resulting criterion has the form of an inequality on the determinant of a matrix associated with an algebraic Riccati equation which depends on a scalar parameter.

The paper is organized as follows. Section II provides the necessary background on the anisotropy of signals and anisotropic norm of systems. Section III establishes ABRL, which constitutes the main result of the paper. Section IV provides an illustrative numerical example. Concluding remarks are given in Section V.

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II. ANISOTROPY OF SIGNALS AND ANISOTROPIC NORM OF SYSTEMS

We now provide a background material on the anisotropy of signals and anisotropic norm of systems. An extended exposition of the anisotropy-based robust performance analysis, developed originally in [9], [10], can be found in [2]; see also [12].

A. Anisotropy of Random Vectors

Recall that for two probability measures P and M on a measurable space $(\Omega, \mathfrak{F})$, the Kullback-Leibler informational divergence (or relative entropy) of P with respect to M is defined by

$$\mathbf{D}(P\|M) \triangleq \begin{cases} \mathbf{E}\ln\frac{dP}{dM} & \text{if } P \ll M \\ +\infty & \text{otherwise} \end{cases}, \qquad (1)$$

where the expectation $\mathbf{E}$ is taken over the measure P, and $dP/dM : \Omega \to \mathbb{R}_+$ is the Radon-Nikodym derivative in the case of absolute continuity of P with respect to M denoted as $P \ll M$; see [1], [3]. The quantity $\mathbf{D}(P\|M)$, which is always nonnegative, is equal to zero if and only if $P = M$. In application of the relative entropy (1) to describing the amount of statistical uncertainty, P is interpreted as the *true* probability measure (which is usually unknown), while M represents its nominal model.

For any $\lambda > 0$, let $p_{m,\lambda}$ denote the Gaussian probability density function (PDF) on $\mathbb{R}^m$ with zero mean and scalar covariance matrix λI_m, so that

$$p_{m,\lambda}(w) \triangleq (2\pi\lambda)^{-m/2}\exp\left(-\frac{|w|^2}{2\lambda}\right), \quad w \in \mathbb{R}^m. \qquad (2)$$

Denote by $\mathbb{L}_2^m$ the class of square integrable $\mathbb{R}^m$-valued random vectors distributed absolutely continuously with respect to the m-dimensional Lebesgue measure mes_m. For any $W \in \mathbb{L}_2^m$ with PDF $f\colon \mathbb{R}^m \to \mathbb{R}_+$, the relative entropy of its distribution with respect to the Gaussian PDF in (2) is computed as

$$\begin{aligned}\mathbf{D}(f\|p_{m,\lambda}) &= \mathbf{E}\ln\frac{f(W)}{p_{m,\lambda}(W)} \\ &= \frac{m}{2}\ln(2\pi\lambda) + \frac{\mathbf{E}(|W|^2)}{2\lambda} - \mathbf{h}(W), \end{aligned} \qquad (3)$$

where

$$\mathbf{h}(W) \triangleq \mathbf{E}\ln f(W) = -\int_{\mathbb{R}^m} f(w)\ln f(w)dw$$

denotes the differential entropy [1] of W with respect to mes_m. The *anisotropy* $\mathbf{A}(W)$ is defined as the minimal value of the relative entropy (3) with respect to the Gaussian distributions in $\mathbb{R}^m$ with zero mean and scalar covariance matrices described by (2):

$$\begin{aligned}\mathbf{A}(W) &\triangleq \min_{\lambda>0}\mathbf{D}(f\|p_{m,\lambda}) \\ &= \frac{m}{2}\ln\left(\frac{2\pi\mathrm{e}}{m}\mathbf{E}(|W|^2)\right) - \mathbf{h}(W), \end{aligned} \qquad (4)$$

where the minimum is achieved at $\lambda = \mathbf{E}(|W|^2)/m$; see [12]. Let $\mathbb{G}^m(\mu, \Sigma)$ denote the class of $\mathbb{R}^m$-valued Gaussian random vectors with mean $\mathbf{E}W = \mu$ and nonsingular covariance matrix $\mathbf{cov}(W) \triangleq \mathbf{E}((W-\mu)(W-\mu)^{\mathrm{T}}) = \Sigma$, so that the corresponding PDF is

$$p(w) \triangleq (2\pi)^{-m/2}(\det\Sigma)^{-1/2}\exp\left(-\frac{1}{2}\|w-\mu\|^2_{\Sigma^{-1}}\right),$$

where $\|w\|_M \triangleq \sqrt{w^{\mathrm{T}}Mw}$ is the Euclidean norm generated by a positive definite matrix M. Basic properties of the anisotropy of a random vector are as follows (see, for example, [12]):

1) The anisotropy $\mathbf{A}(W)$, defined by (4), is invariant under rotation and scaling of W, that is, $\mathbf{A}(\sigma UW) = \mathbf{A}(W)$ for any orthogonal matrix $U \in \mathbb{R}^{m\times m}$ and any $\sigma \in \mathbb{R}\setminus\{0\}$;
2) For any positive definite symmetric matrix $\Sigma \in \mathbb{R}^{m\times m}$,

$$\min\left\{\mathbf{A}(W) : W \in \mathbb{L}_2^m,\ \mathbf{E}(WW^{\mathrm{T}}) = \Sigma\right\} = -\frac{1}{2}\ln\det\frac{m\Sigma}{\mathrm{tr}\,\Sigma}, \qquad (5)$$

where the minimum is only achieved at $W \in \mathbb{G}^m(0,\Sigma)$;
3) $\mathbf{A}(W) \geqslant 0$ for any $W \in \mathbb{L}_2^m$, and $\mathbf{A}(W) = 0$ if and only if $W \in \mathbb{G}^m(0, \lambda I_m)$ for some $\lambda > 0$.

B. Mean Anisotropy of Gaussian Random Sequences

Let $W \triangleq (w_k)_{-\infty<k<+\infty}$ be a stationary sequence of square integrable random vectors with values in $\mathbb{R}^m$ which is interpreted as a discrete-time random signal. Assembling the elements of W, associated with a time interval $[s,t]$, into a random vector

$$W_{s:t} \triangleq \begin{bmatrix} w_s \\ \vdots \\ w_t \end{bmatrix}, \qquad (6)$$

we assume that $W_{0:N}$ is absolutely continuously distributed for every $N \geqslant 0$. The *mean anisotropy* of the sequence W is defined as the anisotropy production rate per time step by

$$\overline{\mathbf{A}}(W) \triangleq \lim_{N\to+\infty}\frac{\mathbf{A}(W_{0:N})}{N}. \qquad (7)$$

It is shown in [12] that

$$\overline{\mathbf{A}}(W) = \mathbf{A}(w_0) + \mathbf{I}(w_0; (w_k)_{k<0}), \qquad (8)$$

where $\mathbf{I}(w_0; (w_k)_{k<0}) \triangleq \lim_{s\to-\infty}\mathbf{I}(w_0; W_{s:-1})$ is the Shannon mutual information [3] between w_0 and the past history $(w_k)_{k<0}$ of the sequence W.

Now suppose the stationary random sequence W is Gaussian. Then

$$\mathbf{I}(w_0; (w_k)_{k<0}) = \frac{1}{2}\ln\det\left(\mathbf{cov}(w_0)\,\mathbf{cov}(\widetilde{w}_0)^{-1}\right), \qquad (9)$$

where

$$\widetilde{w}_0 \triangleq w_0 - \mathbf{E}(w_0 \mid (w_k)_{k<0}) \qquad (10)$$

is the error of the mean-square optimal prediction of w_0 by the past history $(w_k)_{k<0}$ provided by the conditional expectation. Furthermore, let $V \triangleq (v_k)_{-\infty<k<+\infty}$ be an

m-dimensional Gaussian white noise sequence, so that v_k are independent Gaussian random vectors with zero mean $\mathbf{E}v_k = 0$ and identity covariance matrix $\mathbf{cov}(v_k) = I_m$. Suppose $W = GV$ is generated from V by a shaping filter G as

$$w_j = \sum_{k=0}^{+\infty} g_k v_{j-k}, \quad -\infty < j < +\infty. \tag{11}$$

The impulse response of the filter $g_k \in \mathbb{R}^{m\times m}$ is assumed to be square summable over $k \geqslant 0$, thus ensuring the mean square convergence of the series in (11). The spectral density of W is given by

$$S(\omega) \triangleq \widehat{G}(\omega)\widehat{G}(\omega)^*, \quad -\pi \leqslant \omega < \pi, \tag{12}$$

where $(\cdot)^* = \overline{(\cdot)}^{\mathrm{T}}$ denotes the complex conjugate transpose of a matrix, and $\widehat{G}(\omega) \triangleq \lim_{r\to 1-} G(re^{i\omega})$ is the boundary value of the transfer function $G(z) \triangleq \sum_{k=0}^{+\infty} g_k z^k$. The latter encodes all the properties of the filter as an input-output operator and belongs to the Hardy space $\mathcal{H}_2^{m\times m}$ of $(m\times m)$-matrix-valued functions, analytic in the disc $|z| < 1$ of the complex plane. The space is equipped with the $\mathcal{H}_2$-norm using (12) by

$$\|G\|_2 \triangleq \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{tr}\, S(\omega)d\omega}. \tag{13}$$

The covariance matrix of the prediction error (10) and the spectral density (12) are related by the Szegö-Kolmogorov formula [7]

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det S(\omega)d\omega = \ln\det \mathbf{cov}(\widetilde{w}_0). \tag{14}$$

By using (8)–(10), the Szegö limit theorem [4] and (14), it follows that the mean anisotropy (7) of the stationary Gaussian random sequence $W = GV$ can be computed in terms of the spectral density (12) and the associated $\mathcal{H}_2$-norm of the shaping filter (13) as

$$\begin{aligned}\overline{\mathbf{A}}(W) &= -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det \frac{mS(\omega)}{\|G\|_2^2}d\omega \\ &= -\frac{1}{4\pi}\ln\det\frac{m\mathbf{cov}(\widetilde{w}_0)}{\|G\|_2^2};\end{aligned} \tag{15}$$

see [10], [2] for details. Since the probability law of the sequence W is completely determined by the shaping filter G or by the spectral density S, the alternative notations $\overline{\mathbf{A}}(G)$ and $\overline{\mathbf{A}}(S)$ will also be used instead of $\overline{\mathbf{A}}(W)$.

The mean anisotropy functional (15), which is always nonnegative, takes a finite value if the shaping filter G is of full rank, that is, if $\mathrm{rank}\widehat{G}(\omega) = m$ for almost all $\omega \in [-\pi, \pi)$. Otherwise, $\overline{\mathbf{A}}(G) = +\infty$; see [10], [2]. The equality $\overline{\mathbf{A}}(G) = 0$ holds true if and only if G is an all-pass system up to a nonzero constant factor. In this case, the spectral density (12) is described by $S(\omega) = \lambda I_m$, $-\pi \leqslant \omega < \pi$, for some $\lambda > 0$, so that W is a Gaussian white noise sequence with zero mean and a scalar covariance matrix.

C. Anisotropic Norm of Linear Systems

Let $F \in \mathcal{H}_\infty^{p\times m}$ be a system with an m-dimensional input $W = GV$ and a p-dimensional output $Z = FW$, where, as before, V is a m-dimensional Gaussian white noise sequence with zero mean and identity covariance matrix. Let

$$\mathcal{G}_a \triangleq \left\{G \in \mathcal{H}_2^{m\times m} : \overline{\mathbf{A}}(G) \leqslant a\right\} \tag{16}$$

denote the set of shaping filters G which generate Gaussian random sequences W with mean anisotropy (15) bounded by a given parameter $a \geqslant 0$. The *a-anisotropic norm* [10], [2] of the system F is defined by

$$|\!|\!| F |\!|\!|_a \triangleq \sup_{G\in\mathcal{G}_a} \frac{\|FG\|_2}{\|G\|_2}. \tag{17}$$

The fraction on the right-hand side of (17) can also be interpreted as the ratio of the power (semi-) norms [13] of the output $Z = FW$ and input $W = GV$ of the system F. Indeed, by the strong law of large numbers (which involves ergodicity of the signals), and in view of the shorthand notation (6),

$$\lim_{N\to+\infty}\frac{|W_{0:N}|^2}{N} = \|G\|_2^2, \quad \lim_{N\to+\infty}\frac{|Z_{0:N}|^2}{N} = \|FG\|_2^2.$$

Note that, in application to ergodic Gaussian stationary random sequences, the power semi-norm becomes a norm. The quantity $\|FG\|_2/\|G\|_2$, which describes a "stochastic gain" of the system F with respect to $W = GV$, will also be referred to as the *power norm ratio*. The a-anisotropic norm (17) of a given system $F \in \mathcal{H}_\infty^{p\times m}$ is a nondecreasing continuous function of the mean anisotropy level a which satisfies

$$\frac{1}{\sqrt{m}}\|F\|_2 = |\!|\!| F |\!|\!|_0 \leqslant \lim_{a\to+\infty} |\!|\!| F |\!|\!|_a = \|F\|_\infty. \tag{18}$$

These relations show that the $\mathcal{H}_2$ and $\mathcal{H}_\infty$-norms are the limiting cases of the a-anisotropic norm as $a \to 0, +\infty$, respectively.

III. ANISOTROPIC NORM BOUNDED REAL LEMMA

Let $F \in \mathcal{H}_\infty^{p\times m}$ be a LDTI system with an m-dimensional input W, n-dimensional internal state X and p-dimensional output Z governed by

$$\begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x_k \\ w_k \end{bmatrix}, \tag{19}$$

where A, B, C, D are appropriately dimensioned matrices, and A is asymptotically stable (that is, its spectral radius satisfies $\rho(A) < 1$). Suppose W is a stationary Gaussian random sequence whose mean anisotropy does not exceed $a \geqslant 0$. That is, W is generated from the m-dimensional Gaussian white noise V (with zero mean and identity covariance matrix) by an unknown shaping filter G which belongs to the family $\mathcal{G}_a$ defined by (16).

Theorem 1: Let $F \in \mathcal{H}_\infty^{p\times m}$ be a system with the state-space realization (19), where $\rho(A) < 1$. Then the a-anisotropic norm (17) is bounded by a given threshold

$\gamma > 0$, that is, $|\!|\!|F|\!|\!|_a \leqslant \gamma$, if and only if there exists $q \in [0, \min(\gamma^{-2}, \|F\|_\infty^{-2}))$ such that the inequality

$$-\frac{1}{2}\ln\det((1-q\gamma^2)\Sigma) \geqslant a \qquad (20)$$

is satisfied for the matrix Σ associated with the stabilizing ($\rho(A+BL) < 1$) solution $R \succcurlyeq 0$ of the algebraic Riccati equation

$$\begin{aligned} R &= A^{\mathrm{T}}RA + qC^{\mathrm{T}}C + L^{\mathrm{T}}\Sigma^{-1}L, && (21)\\ L &\triangleq \Sigma(B^{\mathrm{T}}RA + qD^{\mathrm{T}}C), && (22)\\ \Sigma &\triangleq (I_m - B^{\mathrm{T}}RB - qD^{\mathrm{T}}D)^{-1}. && (23)\end{aligned}$$

Proof: The power norm ratio $\|FG\|_2/\|G\|_2$ on the right-hand side of (17) and the mean anisotropy $\overline{\mathbf{A}}(G)$ in (15) are both invariant under the scaling of the shaping filter G. Moreover, assuming the system F to be fixed, they are completely specified by the normalized spectral density

$$\Pi(\omega) \triangleq mS(\omega)/\|G\|_2^2 = \frac{2\pi m S(\omega)}{\int_{-\pi}^{\pi}\operatorname{tr} S(v)dv}, \qquad (24)$$

where use has been made of the representation (13). Indeed,

$$\overline{\mathbf{A}}(G) = \alpha(\Pi) \triangleq -\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\det\Pi(\omega)d\omega, \qquad (25)$$

$$\frac{\|FG\|_2}{\|G\|_2} = \nu(\Pi) \triangleq \sqrt{\frac{1}{2\pi m}\int_{-\pi}^{\pi}\operatorname{tr}(\Lambda(\omega)\Pi(\omega))d\omega}, \qquad (26)$$

where the function Π, defined on the interval $[-\pi, \pi)$ by (24), takes values in the set of positive definite Hermitian matrices of order m and satisfies $\int_{-\pi}^{\pi}\operatorname{tr}\Pi(\omega)d\omega = 2\pi m$. Let $\mathbf{\Pi}$ denote the set of normalised spectral densities Π, described above. The system F enters the power norm ratio in (26) only through

$$\Lambda(\omega) \triangleq \widehat{F}(\omega)^*\widehat{F}(\omega). \qquad (27)$$

Note that the squared functional $\nu(\Pi)^2$ is linear, and $\alpha(\Pi)$ is strictly convex with respect to Π. The strict convexity of α follows from the strict concavity of $\ln\det(\cdot)$ considered on the convex cone of positive definite matrices [6]. The strict convexity of the functional α, defined by (25), can also be obtained directly from the positive definiteness of its second variation

$$\begin{aligned}\delta^2\alpha(\Pi) &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\operatorname{tr}(\delta\Pi(\omega)\Pi(\omega)^{-1}\delta\Pi(\omega)\Pi(\omega)^{-1})d\omega\\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\|\Pi(\omega)^{-1/2}\delta\Pi(\omega)\Pi(\omega)^{-1/2}\|^2 d\omega,\end{aligned}$$

where $\delta\Pi$ is the variation of Π, and $\|M\| \triangleq \sqrt{\operatorname{tr}(M^*M)}$ denotes the Frobenius norm of a matrix. Here, we have used the Frechet derivative $d\ln|\det\Sigma|/d\Sigma = \Sigma^{-1}$ and the first variation of the inverse of a nonsingular matrix $\delta(\Sigma^{-1}) = -\Sigma^{-1}(\delta\Sigma)\Sigma^{-1}$. Thus, the minimum value of the mean anisotropy of the disturbance W required to achieve a given level $\gamma > 0$ for the power norm ratio of the system is

$$\begin{aligned}&\min_{\Pi\in\mathbf{\Pi}:\,\nu(\Pi)\geqslant\gamma}\alpha(\Pi)\\ &\quad= -\frac{1}{4\pi}\max_{\Pi\in\mathbf{\Pi}:\,\nu(\Pi)^2\geqslant\gamma^2}\int_{-\pi}^{\pi}\ln\det\Pi(\omega)d\omega\\ &\quad= \min_{0\leqslant q<\|F\|_\infty^{-2}:\,\mathcal{N}(q)\geqslant\gamma}\mathcal{A}(q). \qquad (28)\end{aligned}$$

This is a linearly constrained convex optimization problem. By using the method of Lagrange multipliers, the first minimum in (28) is shown to be achieved at a spectral density which is proportional to

$$S_q(\omega) := (I_m - q\Lambda(\omega))^{-1}, \qquad (29)$$

where q is a subsidiary variable satisfying $0 \leqslant q < \|F\|_\infty^{-2}$. Accordingly, the functions

$$\mathcal{A}(q) \triangleq \alpha(\Pi_q), \quad \mathcal{N}(q) \triangleq \nu(\Pi_q), \qquad (30)$$

are defined by evaluating the functionals α and ν from (25) and (26) at the normalized spectral density

$$\Pi_q(\omega) \triangleq \frac{2\pi m S_q(\omega)}{\int_{-\pi}^{\pi}\operatorname{tr} S_q(v)dv}, \qquad (31)$$

associated with (29) by (24). Now, excluding from consideration the trivial case where the function Λ in (27) is a constant scalar matrix, $\mathcal{A}(q)$ and $\mathcal{N}(q)$ are both strictly increasing in q; see [10], [2] for details. This allows the minimum required mean anisotropy in (28) to be computed as $\mathcal{A}(\mathcal{N}^{-1}(\gamma))$, where $\mathcal{N}^{-1}$ denotes the functional inverse of $\mathcal{N}$. Therefore, the inequality $|\!|\!|F|\!|\!|_a \leqslant \gamma$ is equivalent to $\mathcal{A}(\mathcal{N}^{-1}(\gamma)) \geqslant a$. Now, (29) implies that $\Lambda(\omega) = (I_m - S_q(\omega)^{-1})/q$ and hence,

$$\begin{aligned}&\frac{1}{2\pi m}\int_{-\pi}^{\pi}\operatorname{tr}(\Lambda(\omega)S_q(\omega))d\omega\\ &\qquad= \frac{1}{q}\left(\frac{1}{2\pi m}\int_{-\pi}^{\pi}\operatorname{tr} S_q(\omega)d\omega - 1\right), \qquad (32)\end{aligned}$$

which, in combination with the definition of the function $\mathcal{N}$ via (30), (31) and (26), yields

$$\frac{1}{2\pi m}\int_{-\pi}^{\pi}\operatorname{tr} S_q(\omega)d\omega = 1/(1-q\mathcal{N}(q)^2). \qquad (33)$$

By substituting the last identity into the definition of $\mathcal{A}$ in (30), (31) and (25), it follows that the function can be represented as

$$\mathcal{A}(q) = \mathfrak{A}(q, \mathcal{N}(q)) \qquad (34)$$

in terms of

$$\begin{aligned}\mathfrak{A}(q,\gamma) \triangleq &-\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\det S_q(\omega)d\omega\\ &-\frac{m}{2}\ln(1-q\gamma^2). \qquad (35)\end{aligned}$$

Since $-\ln(1-q\gamma^2)$ is monotonically increasing in $\gamma \in [0, 1/\sqrt{q})$, then so is $\mathfrak{A}(q,\gamma)$. A remarkable property of the function $\mathfrak{A}(q,\gamma)$ is that it achieves its maximum with respect to q at the point $q = \mathcal{N}^{-1}(\gamma)$ where, in view of (34), it coincides with the function $\mathcal{A}$. More precisely,

$$\max_{0\leqslant q<\|F\|_\infty^{-2}} \mathfrak{A}(q,\gamma) = \mathfrak{A}(\mathcal{N}^{-1}(\gamma),\gamma) = \mathcal{A}(\mathcal{N}^{-1}(\gamma)). \quad (36)$$

The significance of this property for establishing a criterion for the inequality $|||F|||_a \leqslant \gamma$ is explained by that (36) implies the equivalence between $\mathcal{A}(\mathcal{N}^{-1}(\gamma)) \geqslant a$ and the existence of $q \in [0, \|F\|_\infty^{-2})$ satisfying $\mathfrak{A}(q,\gamma) \geqslant a$. Therefore,

$$|||F|||_a \leqslant \gamma \Longleftrightarrow \mathfrak{A}(q,\gamma) \geqslant a \text{ for some } q. \quad (37)$$

Now, the property (36) is verified by differentiating the function $\mathfrak{A}$ from (35) with respect to its first argument:

$$\begin{aligned}\partial\mathfrak{A}(q,\gamma)/\partial q &= \frac{1}{4\pi}\int_{-\pi}^{\pi} \partial \ln\det(I_m - q\Lambda(\omega))/\partial q d\omega \\ &\quad + \frac{m\gamma^2}{2(1-q\gamma^2)} \\ &= -\frac{1}{4\pi}\int_{-\pi}^{\pi} \mathrm{tr}(\Lambda(\omega)S_q(\omega))d\omega + \frac{m\gamma^2}{2(1-q\gamma^2)} \\ &\quad - \frac{m\mathcal{N}(q)^2}{2(1-q\mathcal{N}(q)^2)} + \frac{m\gamma^2}{2(1-q\gamma^2)} \\ &= \frac{m(\gamma^2-\mathcal{N}(q)^2)}{2(1-q\gamma^2)(1-q\mathcal{N}(q)^2)}, \end{aligned} \quad (38)$$

where (29), (32) and (33) are used. Since the function $\mathcal{N}$ is strictly monotonic, the representation (38) implies that $\partial\mathfrak{A}(q,\gamma)/\partial q$ is positive for $q < \mathcal{N}^{-1}(\gamma)$ and negative for $q > \mathcal{N}^{-1}(\gamma)$, which indeed establishes (36). It now remains to represent the inequality $\mathfrak{A}(q,\gamma) \geqslant a$ for the function (35) in (37) in terms of the state-space dynamics (19) of the underlying system F. To this end, we note that (29) describes the parametric family of the worst-case spectral densities of the input disturbance W for admissible values of q. Since the subsidiary variable q will be fixed for the rest of the proof, we use the notation

$$S_\star(\omega) \triangleq S_q(\omega) = (I_m - q\Lambda(\omega))^{-1}. \quad (39)$$

We will now obtain a state-space representation of the worst-case disturbance $W_\star$ with the spectral density $S_\star$. In view of (27), the relation (39) is equivalent to

$$\widehat{\Theta}(\omega)^*\widehat{\Theta}(\omega) = I_m, \quad -\pi \leqslant \omega < \pi, \quad (40)$$

where $\widehat{\Theta}$ is the boundary value of the transfer function of the system

$$\Theta \triangleq \begin{bmatrix}\sqrt{q}F\\ G_\star\end{bmatrix}. \quad (41)$$

Here, $G_\star$ is a shaping filter which, in accordance with (12), factorizes the worst-case spectral density (39) as $S_\star = \widehat{G}_\star\widehat{G}_\star^*$. The property (40) means that the system Θ is inner, that is, the ℓ_2-norm of its output coincides with that of the input, as soon as the latter is square summable (note that, except for a trivial zero case, the square summability does not hold for stationary random sequences). Now, the worst-case input disturbance $W_\star = G_\star V$ with the spectral density (39) can be generated as

$$w_k^\star = Lx_k + \sqrt{\Sigma}v_k, \quad (42)$$

where $L \in \mathbb{R}^{m\times n}$ satisfies $\rho(A+BL) < 1$, and $\Sigma \in \mathbb{R}^{m\times m}$ is positive definite. The matrices L and Σ can be found as follows. Substitution of (42) into (19) yields the state-space representation of the worst-case shaping filter

$$G_\star = \left[\begin{array}{c|c} A+BL & B\sqrt{\Sigma}\\ \hline L & \sqrt{\Sigma}\end{array}\right]. \quad (43)$$

Since $\Sigma \succ 0$, then $G_\star$ is invertible, and its inverse is described by

$$G_\star^{-1} = \left[\begin{array}{c|c} A & B\\ \hline -\Sigma^{-1/2}L & \Sigma^{-1/2}\end{array}\right]. \quad (44)$$

Since F and $G_\star^{-1}$ share the matrices A and B, then substitution of (19) and (44) into (41) gives the state-space realization

$$\Theta = \left[\begin{array}{c|c} A & B\\ \hline \Gamma & \Delta\end{array}\right], \quad (45)$$

where the matrices $\Gamma \in \mathbb{R}^{(p+m)\times n}$ and $\Delta \in \mathbb{R}^{(p+m)\times m}$ are defined by

$$\Gamma \triangleq \begin{bmatrix}\sqrt{q}C\\ -\Sigma^{-1/2}L\end{bmatrix}, \quad \Delta \triangleq \begin{bmatrix}\sqrt{q}D\\ \Sigma^{-1/2}\end{bmatrix}. \quad (46)$$

Now, let R denote the observability gramian of the system Θ which is the unique solution of the algebraic Lyapunov equation

$$R = A^{\mathrm{T}}RA + \Gamma^{\mathrm{T}}\Gamma. \quad (47)$$

By applying the state-space criterion of innerness [5], [14] for LDTI systems to (45), it follows that the conditions

$$B^{\mathrm{T}}RA + \Delta^{\mathrm{T}}\Gamma = 0, \quad B^{\mathrm{T}}RB + \Delta^{\mathrm{T}}\Delta = I_m, \quad (48)$$

are sufficient for the system Θ to be inner in sense of (40), and are necessary if the pair (A,B) is controllable. From (46), it follows that

$$\begin{aligned}\Gamma^{\mathrm{T}}\Gamma &= qC^{\mathrm{T}}C + L^{\mathrm{T}}\Sigma^{-1}L,\\ \Delta^{\mathrm{T}}\Gamma &= qD^{\mathrm{T}}C - \Sigma^{-1}L,\\ \Delta^{\mathrm{T}}\Delta &= qD^{\mathrm{T}}D + \Sigma^{-1}.\end{aligned}$$

Hence, substitution of these expressions into (47)–(48) shows that the matrix R is a stabilizing solution to the Riccati equation (21)–(23), with the assumption $\rho(A+BL) < 1$ ensuring the asymptotic stability of the worst-case filter $G_\star$ in (43). Now, since the worst-case input is governed by (42) and V is a white noise sequence with the identity covariance matrix, then the prediction error (10) takes the form $\widetilde{w}_0 = \sqrt{\Sigma}v_0$ and hence, $\mathbf{cov}(\widetilde{w}_0) = \Sigma$. Therefore, in combination with the Szegö-Kolmogorov formula (14), this implies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln\det S_\star(\omega)d\omega = \ln\det\Sigma.$$

Substituting this representation into (35) yields

$$\mathfrak{A}(q,\gamma) = -\frac{1}{2}\ln\det((1-q\gamma^2)\Sigma)$$

and hence, the condition $\mathfrak{A}(q,\gamma) \geqslant a$ is equivalent to the inequality (20) on the matrix Σ associated with the q-dependent Riccati equation (21)–(23). ■

Remark 1: The proof of Theorem 1 shows that its assertion remains valid if the inequality (20) is required to hold as an equality. This, however, does not ensure uniqueness of the pair (q, R) described in the theorem for a given system F and given parameters a and γ, though the stabilizing solutions R of the Riccati equation (21)–(23) are indeed unique for admissible values of the subsidiary variable q, so that there is a well-defined map $q \mapsto R_q$. In fact, the analysis of an auxiliary function $\mathfrak{A}$ in (35), carried out in the proof, shows that the set of those values of q for which the pair (q, R_q) satisfies the inequality (20), form an interval $[q_*, q^*]$ whose endpoints, for a given system F, are functions of a and γ. This interval becomes a singleton $q_* = q^* = \mathcal{N}^{-1}(\gamma)$ if and only if $\gamma = \|\!|F|\!\|_a$. Furthermore, for any $\gamma \geqslant \|\!|F|\!\|_a$, the pair (q_*, R_{q_*}) is a well-defined function of a and γ, which satisfies (20) as an equality, and it is this pair whose behaviour is illustrated in the next section.

Remark 2: From Theorem 1 and its proof it follows that the strict inequality

$$\|\!|F|\!\|_a < \gamma$$

holds if and only if there exists $q \in [0, \min(\gamma^{-2}, \|F\|_\infty^{-2}))$ such that the inequality

$$-\frac{1}{2}\ln\det((1 - q\gamma^2)\Sigma) > a$$

is satisfied for the matrix Σ associated with the stabilizing ($\rho(A + BL) < 1$) solution $R \succ 0$ of the algebraic Riccati equation (21)–(23).

IV. NUMERICAL EXAMPLE

To illustrate application of Theorem 1, we consider an asymptotically stable second-order system F with the state-space realization

$$F = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|cc} 0.01 & 0 & 1 & 0 \\ 0.3 & 0.7 & 0 & 1 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]. \qquad (49)$$

By (18), the a-anisotropic norm $\|\!|F|\!\|_a$ of this system varies from $\|\!|F|\!\|_0 = \|F\|_2/\sqrt{2} = 1.6031$ (for $a = 0$) to $\|F\|_\infty = 4.4454$ (as $a \to +\infty$). Since it makes sense to look for a solution of (20)–(23) only for $\gamma \in [\|\!|F|\!\|_a, \|F\|_\infty]$, we chose the threshold γ, for testing purposes, as $\gamma = (\|\!|F|\!\|_a + \|F\|_\infty)/2$. The Riccati equation (21)–(23) was solved using the Matlab Robust Control Toolbox, with the subsidiary variable q gradually increased (starting from $q = 0$), so that a varied from 0 to 5. The numerical results are presented in Figs. 1 and 2. The test threshold value γ, the anisotropic norm $\|\!|F|\!\|_a$ and the $\mathcal{H}_\infty$-norm $\gamma_\infty = \|F\|_\infty$ are shown in the upper half of Fig. 1. The lower half of Fig. 1 depicts the subsidiary variable q. Fig. 2 shows the eigenvalues of the matrices R (upper diagram) and $A + BL$ (lower diagram) as functions of the mean anisotropy level a. In accordance with positive semi-definiteness of R, its eigenvalues are both nonnegative, and the matrix $A + BL$ remains asymptotically stable, so that R is indeed a stabilizing solution of the algebraic Riccati equation (21)–(23) for all $a \in [0, 5]$. Fig. 3 shows the dependence of the subsidiary variable q and the eigenvalues of the matrices R and $A + BL$ on the parameter γ which varies from γ_a to γ_∞ for a fixed mean anisotropy level a. The results of the test computations demonstrate consistency with predictions of Theorem 1.

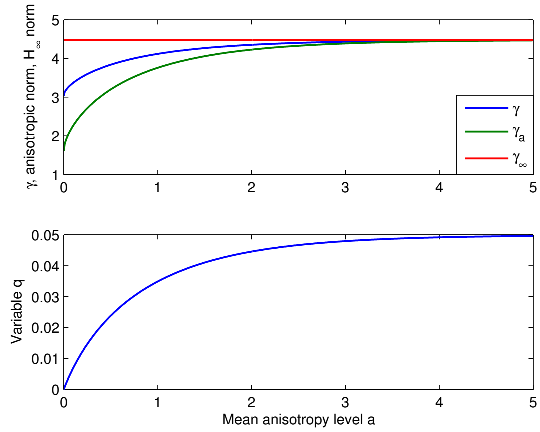

Fig. 1. The a-anisotropic norm $\gamma_a = \|\!|F|\!\|_a$ and the subsidiary variable q as functions of the input mean anisotropy level a for the system (49). Also shown is the $\mathcal{H}_\infty$-norm $\gamma_\infty = \|F\|_\infty$.

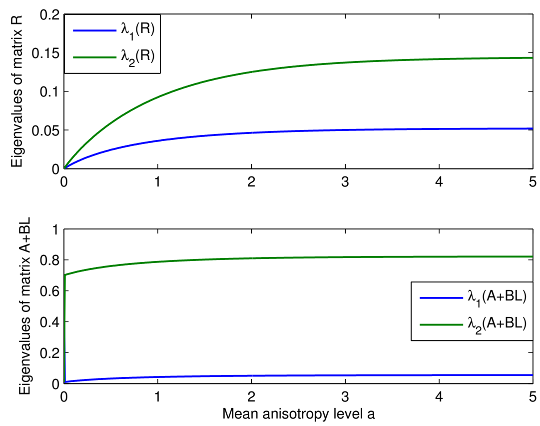

Fig. 2. The eigenvalues of the matrices R and $A + BL$ as functions of the input mean anisotropy level a.

V. CONCLUSION

We have established an Anisotropy-based Bounded Real Lemma (ABRL) which provides a state-space criterion for verifying if the anisotropic norm of a linear discrete-time invariant system is bounded by a given threshold value.

This extends the Bounded Real Lemma of the $\mathcal{H}_\infty$-control theory to stochastic systems where the statistical uncertainty present in the random disturbances is quantified by the mean anisotropy level.

The criterion involves the stabilizing solution of an algebraic Riccati equation, which depends on a subsidiary scalar parameter, and an inequality on the determinant of an associated matrix.

ABRL is applicable to the design of suboptimal controllers

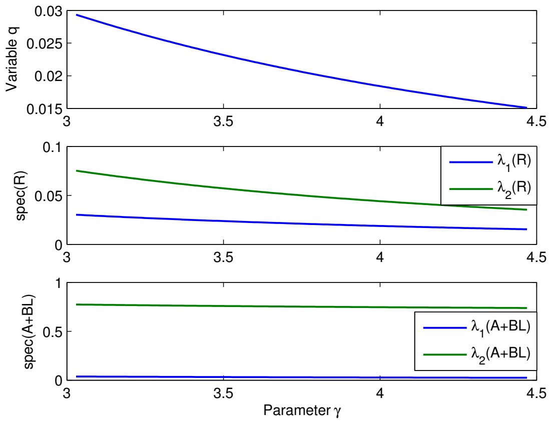

Fig. 3. The subsidiary variable q and the eigenvalues of the matrices R and $A + BL$ as functions of the threshold $\gamma \in [|||F|||_a, \|F\|_\infty]$, computed for the input mean anisotropy level $a = 0.4$.

which ensure a specified upper bound on the anisotropic norm of the closed-loop system, possibly combined with additional specifications which may include a particular pole placement to provide desirable transient characteristics of the system.

REFERENCES

[1] T.M.Cover, and J.A.Thomas. *Elements of Information Theory*, John Wiley and Sons, New York, 1991.

[2] P.Diamond, I.G.Vladimirov, A.P.Kurdyukov, and A.V.Semyonov, Anisotropy-Based Performance Analysis of Linear Discrete Time Invariant Control Systems, *Int. J. of Contr.*, vol. 74, 2001, pp. 28–42.

[3] R.M.Gray, *Entropy and Information Theory*, Springer, New York, 1991.

[4] U.Grenander, and G.Szegö, *Toeplitz Forms and Their Applications*, University of California Press, 1958.

[5] D.-W.Gu, M.C.Tsai, S.D.O'Young, and I.Postlethwaite, State-Space Formulae for Discrete-Time $\mathcal{H}_\infty$ Optimization, *Int J. of Contr.*, vol. 4, 1989, pp. 1683–1723.

[6] R.A.Horn, and C.R.Johnson, *Matrix Analysis*, Cambridge University Press, New York, 2007.

[7] I.A.Ibragimov, and Yu.A.Rozanov, *Gaussian random processes*, Springer-Verlag, New, 1978.

[8] A.V.Semyonov, I.G.Vladimirov, and A.P.Kurdjukov, Stochastic Approach to $\mathcal{H}_\infty$-Optimization, *Proc. 33rd IEEE Conf. on Decision and Control*, Florida, USA, 1994, pp. 2249–2250.

[9] I.G.Vladimirov, A.P.Kurdjukov, and A.V.Semyonov, Anisotropy of Signals and the Entropy of Linear Stationary Systems, *Doklady Math.*, vol. 51, 1995, pp. 388–390.

[10] I.G.Vladimirov, A.P.Kurdjukov, and A.V.Semyonov, On Computing the Anisotropic Norm of Linear Discrete-Time-Invariant Systems, *Proc. 13th IFAC World Congress*, San-Francisco, CA, 1996, pp. 179–184.

[11] I.G.Vladimirov, A.P.Kurdjukov, and A.V.Semyonov, State-space solution to anisotropy-based stochastic $\mathcal{H}_\infty$-optimization problem, *Proc. 13th IFAC World Congress*, San Francisco, USA, 1996, pp. 427–432.

[12] I.G.Vladimirov, P.Diamond, and P.Kloeden, Anisotropy-based performance analysis of finite horizon linear discrete time varying systems, *Automation and Remote Control*, vol. 8, 2006, pp. 1265–1282.

[13] K.Zhou, K.Glover, B.Bodenheimer, and J.Doyle, Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance objectives I: robust performance analysis, *IEEE Trans. AC*, vol. 39, 1994, pp. 1564–1574.

[14] K.Zhou, J.C.Doyle, and K.Glover, *Robust and Optimal Control*, Prentice Hall, Upper Saddle River, NJ, 1996.