Conley Index Theory

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***Abstract*— Conley's index theory provides powerful tools to prove either the existence or the nonexistence of connecting orbits between equilibria of the dynamical systems under consideration. Conley's idea was to relate the local and the global topological properties of the dynamical system by an algebraic object called the connection matrix. The structure of this matrix imposes serious restrictions on the possible configurations of local and global topological data. These restrictions can now be utilized to derive unknown properties of the system out of known ones.**

This proceedings paper is an excerpt from the expository part of [BMP09], where the interested reader can find advanced applications of CONLEY index theory.

I. HISTORICAL REMARKS

In order to give some examples illustrating the usefulness of CONLEY index theory we first make some historical remarks. This introductory section is not necessary to understand the formal theory starting in Section II.

One of the motivations of CONLEY was to generalize MORSE theory. The main idea of MORSE theory is to study the topological properties of an n-dimensional smooth manifold X by studying the critical points of a so-called MORSE function $f : X \to \mathbb{R}$. A MORSE function is a smooth function with non-degenerate critical points. For such a non-degenerate critical point $p \in X$ the MORSE lemma guarantees the existence of a coordinate system $x = (x_1, \dots, x_n)$ around p such that f can be written as

$$f(x) = f(p) - x_1^2 - \cdots - x_\gamma^2 + x_{\gamma+1}^2 + \cdots + x_n^2.$$

The number γ is called the MORSE index of the critical point p and denoted by $\operatorname{index}(p)$.

Let c_γ denote the number of critical points of MORSE index γ. The MORSE formula

$$\chi(X) = \sum_{\gamma=0}^{n} (-1)^\gamma c_\gamma. \quad (1)$$

computes the EULER-POINCARÉ characteristic $\chi(X)$ in terms of the indices of the critical points of f, where by definition

$$\chi(X) := \sum_{i=0}^{n} (-1)^i \beta_i(X),$$

which is the alternating sum of the BETTI numbers.

We illustrate this with two examples in Figure 1. It turns out that the height function $f(x) = \operatorname{height}(x)$ is a MORSE

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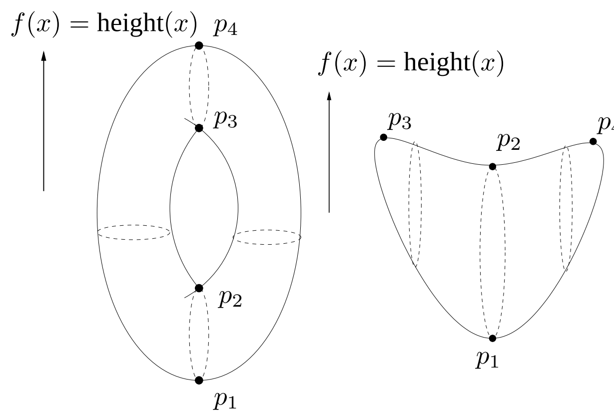

Fig. 1. Critical points on torus $\mathbb{T}$ and heart $\mathbb{H}$

function for both surfaces. On the torus $\mathbb{T}$ we have four critical points $p_1, \dots, p_4$ with $c_0 = c_2 = 1$ and $c_1 = 2$. On the heart $\mathbb{H}$ we have also four critical points but $c_0 = c_1 = 1$ and $c_2 = 2$. Hence $\chi(\mathbb{T}) = 0$ and $\chi(\mathbb{H}) = 2$.

Now CONLEY's idea was to replace f by the flow generated by the gradient ∇f and develop a general index theory for flows on manifolds. Note that in CONLEY's theory the flow is not necessarily a gradient flow.

The first central notion in CONLEY index theory is the CONLEY index of isolated invariant sets (a formal definition will be given in (4)). For the purpose of this introductory section we only need the CONLEY index of the whole manifold X and those of the hyperbolic equilibria p of the flow, which correspond to the (non-degenerate) critical points of the MORSE function f.

Define the *homology* CONLEY *index* $CH_*(X)$ of the closed manifold X as the graded object of homology groups of X, i.e.

$$CH_*(X) := H_*(X) = H_*(X, \emptyset) = (H_0(X), H_1(X), \dots). \quad (2)$$

For the purpose of this section we take homology with values in a field $\mathbb{K}$ for simplicity.

Let γ denote the dimension of the unstable manifold of a hyperbolic equilibrium $p \in X$. If the flow φ is a gradient flow of a MORSE function f, then γ is the MORSE index $\operatorname{index}(p)$ mentioned above. The homology CONLEY index $CH_*(p)$ of p now generalizes the MORSE index in the following way (see also Prop. 2.1): It is again a graded object of $\mathbb{K}$-vector spaces $CH_*(p) = (CH_0(p), CH_1(p), \dots)$ such that

$$CH_i(p) \cong \begin{cases} \mathbb{K} & \text{if } i = \gamma\ , \\ 0 & \text{otherwise}\ . \end{cases}$$

The second central notion in CONLEY index theory is that of a *connection matrix*.

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We assume for simplicity that φ is a strict gradient flow with a finite set P of equilibria, all of them hyperbolic. Define the sum of graded objects

$$C = (C_i)_{i\geq 0} = \bigoplus_{p\in P} CH_*(p).$$

Consider a sequence of maps $\Delta = (\Delta_1, \Delta_2, \ldots)$ with $\Delta_i : C_i \to C_{i-1}$ such that $\Delta_{i-1} \circ \Delta_i = 0$, turning C into a complex. We will write C^Δ for the complex C endowed with Δ as a boundary operator. A sequence Δ is called a connection matrix if, among other things (see Def. 2.5), the following property holds:

$$H_i(C^\Delta) \cong CH_i(X). \tag{3}$$

In our heart example from above it can be verified that $\Delta = (\Delta_1, \Delta_2)$ in

$$C^\Delta : 0 \leftarrow \mathbb{K}^1 \xleftarrow{\Delta_1 = \begin{pmatrix} 0 \end{pmatrix}} \mathbb{K}^1 \xleftarrow{\Delta_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}} \mathbb{K}^2 \leftarrow 0$$

is a connection matrix.

One of the main results of CONLEY index theory implies that non-trivial entries in this connection matrix correspond to *heteroclinic connections* between $p_3 \to p_2$ and $p_4 \to p_2$. Another major result is FRANZOSA's existence result of a connection matrix [Fra89, Thm. 3.8] yielding in particular (3), whereas uniqueness in not always guaranteed.

Therefore, in general, connection matrices may be used to reduce the huge amount of possible heteroclinic connections, and even prove existence of some of the connections.

As a nice application of the above developed notion, the MORSE formula (1) now immediately follows from the existence of a connection matrix. In case φ is the gradient flow of a MORSE function f then $\dim_\mathbb{K} C_i^\Delta = c_i$, the number of critical points of f with MORSE index i. Then

$$\begin{aligned}
\chi(X) &= \sum_{i=0}^n (-1)^i \dim_\mathbb{K} H_i(X) \\
&\overset{(2)}{=} \sum_{i=0}^n (-1)^i \dim_\mathbb{K} CH_i(X) \\
&\overset{(3)}{=} \sum_{i=0}^n (-1)^i \dim_\mathbb{K} H_i(C^\Delta) \\
&= \sum_{i=0}^n (-1)^i \dim_\mathbb{K} C_i^\Delta \\
&= \sum_{i=0}^n (-1)^i c_i,
\end{aligned}$$

where the fore-last equation is a standard application of the homomorphism theorem.

II. CONLEY INDEX THEORY

Let X be a locally compact metric space. The object of study is a *flow* $\varphi : \mathbb{R} \times X \to X$, i.e., a continuous map $\mathbb{R} \times X \to X$ which satisfies $\varphi(0, x) = x$ and $\varphi(s, \varphi(t, x)) = \varphi(s + t, x)$ for all $x \in X$ and $s, t \in \mathbb{R}$. (X, φ) is called a *dynamical system*.

A. *Homology* CONLEY *index*

The following theory has been initiated by CONLEY [Con78] in order to study invariant sets of dynamical systems. For a subset $Y \subset X$ define

$$\operatorname{Inv}(Y) := \operatorname{Inv}(Y, \varphi) := \{x \in Y \mid \varphi(\mathbb{R}, x) \subset Y\} \subset Y,$$

the *invariant subset* of Y.

A subset $S \subset X$ is *invariant* under the flow φ, if $S = \operatorname{Inv}(S)$. S is called an *isolated invariant set* if there exists a *compact* set $Y \subset X$ (an *isolating neighborhood*) such that

$$S = \operatorname{Inv}(Y) \subset \overset{\circ}{Y},$$

where $\overset{\circ}{Y}$ denotes the interior of Y.

Let M be an isolated invariant set. A pair of compact sets (N, L) with $L \subset N$ is called an *index pair* for M (cf. [MM02, Def. 2.4]) if

1) $\overline{N \setminus L}$ is an *isolating neighborhood* of M.
2) L is *positively invariant*, i.e., $\varphi([0, t], x) \subset L$ for all $x \in L$ satisfying $\varphi([0, t], x) \subset N$.
3) L is an *exit set* for N, i.e., for all $x \in N$ and all $t_1 > 0$ such that $\varphi(t_1, x) \notin N$, there exists a $t_0 \in [0, t_1]$ for which $\varphi([0, t_0], x) \subset N$ and $\varphi(t_0, x) \in L$.

Let $M \subset S$ be an isolated invariant set with index pair (N, L). We associate to such a pair a complex $\mathcal{C}_*(N, L) \cong \mathcal{C}_*(N)/\mathcal{C}_*(L)$ of relative (simplicial or singular ...) chains. The *homology* CONLEY *index* of M is defined by

$$CH_*(M) = H_*(N, L) := H_*(\mathcal{C}_*(N, L)), \tag{4}$$

where $H_*(N, L) = (H_k(N, L))_{k\in\mathbb{Z}_{\geq 0}}$ denotes the relative homology groups (cf. [MM02, Def. 3.7, Thm. 3.8]). Note, that there always exits an index pair (N, L), such that $H_*(N, L) = H_*(N/L, [L])$ (see [MM02, Remark 3.9]). We usually take coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Before we proceed, let us recall the homology CONLEY index of some specific isolated invariant sets.

Proposition 2.1: Assume that S contains only a hyperbolic fixed point with an unstable manifold of dimension n (i.e., MORSE index n). Then S is an isolated invariant set and

$$CH_k(S) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = n\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

For the remainder of this paper, we will abbreviate this statement by saying that the CONLEY index of S is equal to Σ^n, i.e., write $CH_*(S) = \Sigma^n$.

Note, that usually $\Sigma^n = (S^n, *)$ denotes the *homotopy* type of the index pair (N, L) of a hyperbolic fixed point of MORSE index n. But since we are only interested in homology, we abuse the notation.

In order to apply CONLEY's theory to MORSE decompositions of the attractor, we need to know the CONLEY index of the attractor itself. By the continuation property [MM02, Thm. 3.10] it is the same as the one of a stable fixed point (see [MM96, Prop. 4.1]).

Proposition 2.2: If the dynamical system (X, φ) possesses a global attractor $\mathcal{A}$, then we have

$$CH_k(\mathcal{A}) = \begin{cases} \mathbb{Z}_2 & \text{for} \quad k = 0\,, \\ 0 & \text{for} \quad k \neq 0\,. \end{cases}$$

The empty set S is also an isolated invariant set having the trivial CONLEY index

$$CH_k(S) = 0 \quad \text{for all } k. \tag{5}$$

This occurs e.g. if an isolating neighborhood of a parallel flow is considered. Similarly, a heteroclinic connection between two hyperbolic fixed points stemming from a saddle node bifurcation has trivial CONLEY index, although the isolated invariant set is no longer empty.

B. Posets

A set P together with a strict partial order $>$ (i.e., an irreflexive and transitive relation $> \subset P \times P$) is called a *poset* (i.e., partially ordered set) and is denoted by $(P, >)$.

A subset $I \subset P$ is called an *interval* in $(P, >)$ if for all $p, q \in I$ and $r \in P$ the following implication holds:

$$q > r > p \quad \Rightarrow \quad r \in I.$$

The set of all intervals in $(P, >)$ is denoted by $\mathcal{I}(P, >)$.

An n-tuple $(I_1, \ldots, I_n)$, $n \geq 2$, of intervals in $(P, >)$ is called *adjacent* if these intervals are mutually disjoint, $\bigcup_{i=1}^n I_i$ is an interval in $(P, >)$ and for all $p \in I_j$, $q \in I_k$ the following implication holds:

$$j < k \quad \Rightarrow \quad p \not> q.$$

The set of all adjacent n-tuples of intervals in $(P, >)$ is denoted by $\mathcal{I}_n(P, >)$.

If $(I_1, \ldots, I_n)$ is an adjacent n-tuple of intervals in $(P, >)$, then denote $I_1 I_2 \ldots I_n := \bigcup_{i=1}^n I_i$, which by definition is again an interval.

If $(I, J) \in \mathcal{I}_2(P, >)$ as well as $(J, I) \in \mathcal{I}_2(P, >)$, then I and J are said to be *incomparable*.

C. MORSE *decomposition*

For a subset $Y \subset X$ the *ω-limit set* of Y is $\omega(Y) := \bigcap_{t>0} \overline{\varphi([t, \infty), Y)}$, while the *$\alpha$-limit set* of Y is $\alpha(Y) := \bigcap_{t>0} \overline{\varphi((-\infty, -t), Y)}$.

For two subsets $Y_1, Y_2 \subset X$ define the *set of connecting orbits*

$$\begin{aligned} \mathrm{Con}(Y_1, Y_2) &= \mathrm{Con}(Y_1, Y_2; X) \\ &:= \{x \in X \mid \alpha(x) \subset Y_1 \text{ and } \omega(x) \subset Y_2\}. \end{aligned}$$

Let S be an isolated invariant set and $(P, >)$ be a poset. A finite collection

$$\mathcal{M}(S) = \{M(p) \mid p \in P\}$$

of disjoint isolated invariant subsets $M(p)$ of S is called a MORSE *decomposition* if there exists a strict partial order $>$ on P, such that for every $x \in S \setminus \bigcup_{p \in P} M(p)$ there exist $p, q \in P$, such that $q > p$ and $x \in \mathrm{Con}(M(q), M(p))$.

The sets $M(p)$ are called MORSE *sets*. A partial order on P satisfying this property is said to be *admissible*.

There is a partial order $>_\varphi$ induced by the flow, generated by the relations $q >_\varphi p$ whenever $\mathrm{Con}(M(q), M(p)) \neq \emptyset$. This so called *flow-induced order* is a subset of every admissible order, and in this sense minimal. Normally this order is not known and one is content with a coarser order. If, for example, a LYAPUNOV or energy function E is known with $E(x) > E(\varphi(t, x))$ for all $t > 0$, whenever $x \in X$ is not a steady state, then defining the partial order $>_E$ by

$$q >_E p, \text{ iff } E(y) > E(x) \text{ for all } y \in M(q) \text{ and } x \in M(p),$$

yields an admissible order, in case the energy levels of all non-equilibrium MORSE sets are isolated in the energy spectrum. This order is called *energy-induced order*.

For an interval I define the set

$$M(I) := \bigcup_{p \in I} M(p) \cup \bigcup_{p,q \in I} \mathrm{Con}(M(q), M(p)).$$

The set $M(I)$ is again an isolated invariant set (cf. [MM02, Prop. 2.12]). If $(I, J) \in \mathcal{I}_2(P, >)$, then $(M(I), M(J))$ is an *attractor-repeller pair* in $M(IJ)$ (cf. [MM02, Def. 2.1]).

D. Connection matrices

We start by revising the definition of connection matrices following [BR09]. Contrary to [BR09] we apply matrices from the left and hence use the column convention, as widely used in the CONLEY index literature.

Let $\mathcal{M}(S) = \{M(p) \mid p \in (P, >)\}$ be a MORSE decomposition of S. Hence each $M(p)$ is an isolated invariant set and the CONLEY index $CH_*(M(p))$ is well-defined (by (4)). In what follows, we consider the collection $C := \{CH_*(M(p)) \mid p \in P\}$ of abelian groups, which are indexed by P, and a group homomorphism

$$\Delta : \bigoplus_{p \in P} CH_*(M(p)) \to \bigoplus_{p \in P} CH_*(M(p)). \tag{6}$$

For an interval I in $(P, >)$ set

$$C_*(I) := \bigoplus_{p \in I} CH_*(M(p))$$

and denote by $\Delta(I) : C_*(I) \to C_*(I)$ the homomorphism $\pi_I \circ \Delta \circ \iota_I$, where $\iota_I : C_*(I) \to C_*(P)$ is the canonical injection and $\pi_I : C_*(P) \to C_*(I)$ is the canonical projection.

If $p_1, p_2 \in P$, we refer to the restriction of Δ to $C_*(p_2)$ by $\Delta(\cdot, p_2) : C_*(p_2) \to C_*(P)$, and the composition $\pi_{p_1} \circ \Delta(\cdot, p_2)$, where π_{p_1} is the projection $C_*(P) \to C_*(p_1)$, is denoted by $\Delta(p_1, p_2) : C_*(p_2) \to C_*(p_1)$. Then Δ can be visualized as a matrix with $\Delta(\cdot, p_2)$ as its p_2-th column and $\Delta(p_1, p_2)$ as its entry at position (p_1, p_2). In particular, for $I \in \mathcal{I}(P, >)$ the homomorphism $\Delta(I)$ may be be represented as

$$\begin{aligned} \Delta(I) = (\Delta(p_1, p_2))_{p_1, p_2 \in I} : \\ \bigoplus_{p \in I} CH_*(M(p)) \to \bigoplus_{p \in I} CH_*(M(p)). \end{aligned}$$

Definition 2.3 ([Fra88, Def. 1.3]): Δ being as above:

1) Δ is said to be *upper triangular* if $\Delta(p_1, p_2) \neq 0$ implies $p_2 > p_1$ or $p_1 = p_2$.
2) Δ is said to be *strictly upper triangular* if $\Delta(p_1, p_2) \neq 0$ implies $p_2 > p_1$.
3) Δ is called a *boundary map* if it is a homomorphism of degree -1, i.e., it maps $C_n(P)$ to $C_{n-1}(P)$, and $\Delta \circ \Delta = 0$.

Proposition 2.4 ([Fra89, Prop. 3.3]):
Let $C = \{CH_*(M(p)) \mid p \in P\}$ be as above and let $\Delta : \bigoplus_{p\in P} CH_*(M(p)) \to \bigoplus_{p\in P} CH_*(M(p))$ be an upper triangular boundary map. Then:

1) $C_*(I)$ and $\Delta(I)$ form a chain complex denoted by $C_*^\Delta(I)$ for all $I \in \mathcal{I}(P, >)$.
2) For all $(I, J) \in \mathcal{I}_2(P, >)$, the obvious injection and projection maps $i(I, IJ)$ and $p(IJ, J)$ are chain maps and
$$0 \to C_*^\Delta(I) \xrightarrow{i(I,IJ)} C_*^\Delta(IJ) \xrightarrow{p(IJ,J)} C_*^\Delta(J) \to 0 \qquad (7)$$
is a short exact sequence.

In other words, the degree -1 property and $\Delta \circ \Delta = 0$ endow $C_*(P)$ with a chain complex structure (called $C_*^\Delta(P)$). The property "upper triangular" guarantees that $\Delta(I)$ is also a boundary operator on $C_*(I)$ leading to $C_*^\Delta(I)$. It further implies for a pair (I, J) of adjacent intervals that $\Delta(IJ)_{|C_*(I)} = \Delta(I)$, allowing one to view $C_*^\Delta(I)$ as a subcomplex of $C_*^\Delta(IJ)$, with $C_*^\Delta(J)$ being naturally isomorphic to the quotient complex $C_*^\Delta(IJ)/C_*^\Delta(I)$, making (7) a short exact sequence.

The first statement of Proposition 2.4 allows one to define the homology groups
$$H_*(C_*^\Delta(I)) := \ker \Delta(I)/\mathrm{im}\Delta(I),$$
shortly denoted as $H_*\Delta(I)$, while the second statement leads for each $(I, J) \in \mathcal{I}_2(P, >)$ to a long exact homology sequence
$$\cdots \longrightarrow H_{n+1}\Delta(J) \xrightarrow{\delta_{n+1}} H_n\Delta(I) \longrightarrow H_n\Delta(IJ) \longrightarrow H_n\Delta(J) \xrightarrow{\delta_n} H_{n-1}\Delta(I) \longrightarrow \cdots, \qquad (8)$$
where δ_* are the connecting homomorphisms constructed by the snake Lemma.

To state the definition of a connection matrix we still need some more preliminaries from the dynamics side. For a pair (I, J) of adjacent intervals $(M(I), M(J))$ is an attractor-repeller pair for the isolated invariant set $M(IJ)$, as stated before.

By definition an *index triple* (N_2, N_1, N_0) for the attractor-repeller pair $(M(I), M(J))$ satisfies $N_0 \subset N_1 \subset N_2$ and

(i) (N_2, N_0) is an index pair for the isolated invariant set $M(IJ)$;
(ii) (N_2, N_1) is an index pair for the repeller $M(J)$;
(iii) (N_1, N_0) is an index pair for the attractor $M(I)$.

The existence of an index triple (N_2, N_1, N_0) for the attractor-repeller pair $(M(I), M(J))$ is always guaranteed (cf. [MM02, Thm. 4.2]), providing a short exact sequence of chain complexes
$$0 \to \mathcal{C}_*(N_1, N_0) \to \mathcal{C}_*(N_2, N_0) \to \mathcal{C}_*(N_2, N_1) \to 0,$$
where $\mathcal{C}_*(N_i, N_j)$ is the complex of relative chains as in (4). This short exact sequence induces a long exact homology sequence
$$\cdots \longrightarrow H_{n+1}(N_2, N_1) \xrightarrow{\partial_{n+1}} H_n(N_1, N_0) \longrightarrow H_n(N_2, N_0) \longrightarrow H_n(N_2, N_1) \xrightarrow{\partial_n} H_{n-1}(N_1, N_0) \longrightarrow \cdots.$$

In other words the last long exact sequence is by definition (cf. (4))
$$\cdots \longrightarrow CH_{n+1}(M(J)) \xrightarrow{\partial_{n+1}} CH_n(M(I)) \longrightarrow CH_n(M(IJ)) \longrightarrow CH_n(M(J)) \xrightarrow{\partial_n} CH_{n-1}(M(I)) \longrightarrow \cdots. \qquad (9)$$

Now we are ready to state the definition of a connection matrix (cf. [Fra89, Def. 3.6]), which avoids braids (cf. [BR09, Def. 2.7]). The definition of a connection matrix relates the algebraically induced long exact sequence (8) and the dynamically induced long exact sequence (9), more precisely:

Definition 2.5 (Connection matrix):
Let $C = \{CH_*(M(p)) \mid p \in P\}$ be as above and let $\Delta : \bigoplus_{p\in P} CH_*(M(p)) \to \bigoplus_{p\in P} CH_*(M(p))$ be an upper triangular boundary map. Δ is called a *connection matrix* if for each interval $K \in \mathcal{I}(P, >)$ there exits an isomorphism $\theta(K) : H_*\Delta(K) \to CH_*(M(K))$ such that for all pairs

$(I, J) \in \mathcal{I}_2(P, >)$ of adjacent intervals the following diagram

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow{\scriptstyle \delta_{n+1}} & & \downarrow{\scriptstyle \partial_{n+1}} \\ H_n\Delta(I) & \xrightarrow{\theta(I)} & CH_n(M(I)) \\ \downarrow & & \downarrow \\ H_n\Delta(IJ) & \xrightarrow{\theta(IJ)} & CH_n(M(IJ)) \\ \downarrow & & \downarrow \\ H_n\Delta(J) & \xrightarrow{\theta(J)} & CH_n(M(J)) \\ \downarrow{\scriptstyle \delta_n} & & \downarrow{\scriptstyle \partial_n} \\ H_{n-1}\Delta(I) & \xrightarrow{\theta(I)} & CH_{n-1}(M(I)) \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array} \tag{10}$$

is an isomorphism of long exact sequences, i.e., that additionally all the squares commute.

Remark 2.6 ([BR09, Remark 3.4]): We want to emphasize the importance of first choosing a fixed isomorphism $\theta(K)$ for each interval K. This single isomorphism enters in all the commutative diagrams (10). Notably, in FRANZOSA's definition of connection matrices also a fixed isomorphism $\theta(K)$ for each interval K has to be chosen *a priori* (cf. [Fra88, Def. 1.2] or [Fra89, Def. 2.4]).

Following [BR09], we show that this braid free definition coincides with Franzosa's definition of connection matrices [Fra88, Def. 1.4] or [Fra89, Def. 3.6]:
In contrast to the above definition of connection matrices, FRANZOSA's definition requires the isomorphism of two graded module braids. The first braid is obtained as the homology of a chain complex braid in the setup of the upper triangular boundary map Δ (cf. [Fra89, Prop. 3.4] together with [Fra89, Prop. 2.7]). The other is obtained as the homology of the chain complex braid of an index filtration, which in turn generalizes our index triples. SALAMON proved in [Sal85] that index filtrations of MORSE decompositions always exit, (see also [Fra86, Thm. 3.8], [FM88], and [Mis95, Thm. 4.2.4]). That an index filtration of a MORSE decomposition always induces a chain complex braid was proved in [Fra86, Section 4], see also the discussion before [Mis95, Def. 4.3.2].

Clearly, and because of the a priori chosen isomorphisms $\theta(K)$, the isomorphism of the long exact sequences in Definition 2.5 gives rise to the isomorphism of the graded module braids, as required by FRANZOSA.

Corollary 2.7: The definition of connection matrices following FRANZOSA [Fra88, Def. 1.4] is equivalent to Definition 2.5 above.

FRANZOSA's existence theorem [Fra89, Thm. 3.8] guarantees the existence of at least one connection matrix, provided all $CH_*(M(p))$ are free over the coefficient ring. By taking coefficients in a field, as we do by taking $\mathbb{Z}/2\mathbb{Z}$-coefficients, this is immediate.

Remark 2.8: In practice, the lack of topological data, in particular of the index triples on the dynamical side, prevents us from constructing the maps in (9) explicitly. Therefore in the software package `conley`, we can only check a part of the defining properties of connection matrices. More precisely, besides Δ being an upper triangular boundary map [MPMW07, Section 3, (C1,C2)], we, so far, only check abstract isomorphisms $H_n\Delta(K) \cong CH_n(M(K))$ for each interval $K \in \mathcal{I}(P, >)$ as in [MPMW07, Section 3, (C3)]. Accordingly, we call the matrices computed by `conley` *possible* connection matrices.

III. EXAMPLE

A. FRANZOSA's *transition matrix example*

We look at a gradient flow φ serving as a transition system connecting the two systems

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \theta y - x\left(x - \frac{1}{3}\right)\left(1 - x\right). \end{aligned} \tag{11}$$

at $\theta = \theta'$ and $\theta = \theta''$:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \theta y - x\left(x - \frac{1}{3}\right)(1 - x) \\ \dot{\theta} &= \varepsilon(\theta' - \theta)(\theta'' - \theta), \end{aligned} \tag{12}$$

with $0 < \theta' \ll 1$ and $1 \ll \theta'' < \infty$ as before and small $\varepsilon > 0$ fixed. This is also studied in [Fra89, Example 6.2]. The additional equation for $\theta = \theta(t)$ is decoupled from the others. $\{\theta = \theta'\}$ is invariant and attracting, while $\{\theta = \theta''\}$ is invariant and repelling (cf. Figure III-A). A sketch of the combined flow is given in Figure III-A.

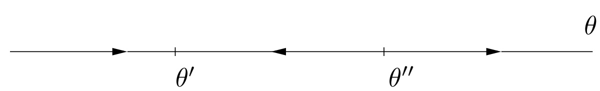

Fig. 2. The flow in the θ-component

We refer for coding details to the example worksheet `Franzosa_all` on the homepage [BMPR08].

Let $1', 2', 3'$ denote the equilibria for $\theta = \theta'$ and $1'', 2'', 3''$ denote those for $\theta = \theta''$. $1', 2', 3'$ retain their CONLEY indices, while the CONLEY indices of $1'', 2'', 3''$ are raised by one, i.e.

$$\begin{array}{rclrcl} CH_*(M(1')) &=& \Sigma^0, & CH_*(M(2')) &=& \Sigma^1, \text{ and} \\ CH_*(M(3')) &=& \Sigma^1; & & & \\ CH_*(M(1'')) &=& \Sigma^1, & CH_*(M(2'')) &=& \Sigma^2, \text{ and} \\ CH_*(M(3'')) &=& \Sigma^2. & & & \end{array} \tag{13}$$

In this worksheet we perform computations with different orders and different CONLEY index data.

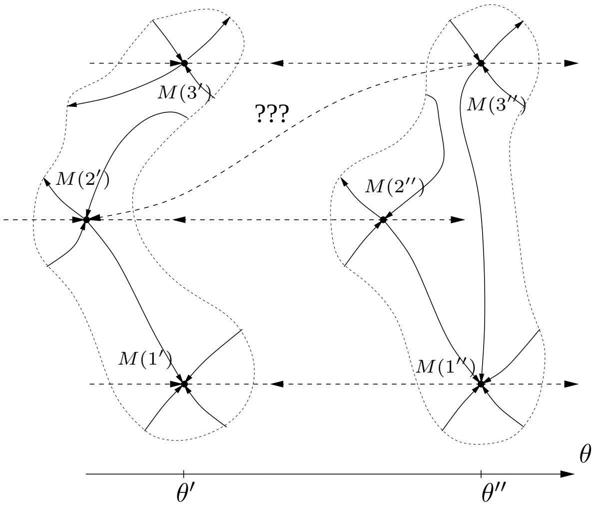

Fig. 3. The flow φ of the extended system

Variant 1: The flow-induced order $>_\varphi$ (in short $>$) at least contains all connections known for the $\theta = \theta'$ and the $\theta = \theta''$ system, together with the connections between the different copies of the equilibria in the two θ-systems, i.e., we have at least the relations

$$\begin{aligned} 2' > 1', \quad 2'' > 1'', \quad 3'' > 1'', \\ 1'' > 1', \quad 2'' > 2', \quad 3'' > 3'. \end{aligned} \tag{14}$$

To the CONLEY indices of the equilibria we only add the CONLEY index of the whole invariant set $M(P)$, which is trivial, i.e., besides (13)

$$CH_*(M(P)) = 0. \tag{15}$$

Apparently we get non–unique connection matrices

$$\begin{bmatrix} \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

$$\begin{bmatrix} \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

where each 0 is replaced by a dot. Maybe we didn't provide enough CONLEY index data?

Variant 2: We retain the proposed order above, but add CONLEY index information about the "transition"-interval $\{1', 1''\}$ and the θ''-interval $\{1'', 3''\}$. By construction, there is a heteroclinic connection from $i'' \to i'$ with CONLEY index $CH_*(M(\{i', i''\})) = 0$, i.e., the CONLEY data we use additionally to (13) and (15) is

$$CH_*(M(\{1', 1''\})) = CH_*(M(\{1'', 3''\})) = 0. \tag{16}$$

Explicit computations in the worksheet show that there are no connection matrices satisfying the above requirements. But due to FRANZOSA's existence result [Fra89, Thm. 3.8] at least one connection matrix is always guaranteed. This inconsistency tells us that our proposed order is not admissible, i.e., that it does not contain the flow-induced order.

The strategy now is to enlarge the order, as a subset of $P \times P$, to avoid inconsistency. The following four possibilities $2'' > 3'$, $3'' > 2'$, $1'' > 2'$, $1'' > 3'$ are in question. The last two can be ruled out immediately, because adding $1'' > 2'$ implies that $\{2', 2''\}$ (and $\{1', 1''\}$) is not anymore an interval, and adding $1'' > 3'$ implies that $\{3', 3''\}$ is no longer an interval. However, the sets $\{i', i''\}$ are always intervals by construction.

Variant 3: We add $2'' > 3'$ to the generating relations and retain the enriched CONLEY index information above (13), (15), and (16). Again we run into an inconsistency (no connection matrix matches the data), which tells us that our proposed order is again not admissible, i.e., that it does not contain the flow induced order.

Variant 4: The only possible enlargement left is the relation $3'' > 2'$. This indeed proves that the flow-induced order $>_\varphi$ is generated by the relations (14) together with $3'' > 2'$. Our computations yield the unique connection matrix

$$\begin{bmatrix} \cdot & 1 & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

Thus, the above inconsistency is resolved. Since furthermore $(\{2'\}, \{3''\}) \in \mathcal{I}_2(P, >_\varphi)$ is a pair of adjacent intervals and $\Delta(2', 3'') \neq 0$ (cf. [MPMW07, Section 3,(C4)]), the existence of a connecting orbit $3'' \to 2'$ is proved. This connecting orbit was already found in FRANZOSA's original article (see [Fra89, Example 6.2]). The line of arguments provided above is nevertheless new.

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